PROBLEM SET TWO: DUE FRIDAY, SEPTEMBER 18

Problem 1. Let A be a totally nonnegative $n \times n$ matrix of rank n. Show that the diagonal entries A_{kk} are positive. Hint: You should only need the positivity of the 1×1 and 2×2 minors.

Problem 2. Let A be an $m \times n$ real matrix, which we can think of as giving a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Then, functorially, we get a map $\bigwedge^k(A) : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^m$.

(1) Show that A is totally positive iff and only if all entries of the matrix of $\bigwedge^k(A)$ in the "obvious" basis are positive.
(2) Let A be a totally positive $n \times n$ matrix. Let $\mu_1, \mu_2, \ldots, \mu_{\binom{n}{k}}$ be the complex roots of the characteristic polynomial of $\bigwedge^k(A)$. Show that there is a positive real μ_j, which has $\mu_j > |\mu_r|$ for all $r \neq j$. (Hint: Look up the Perron-Frobenius theorem; there isn't much to do here.)
(3) Let A be a totally positive $n \times n$ matrix. Show that A has n distinct positive real eigenvalues.

Problem 3. We reuse the notations s_i and $\ell(\sigma)$ from the previous problem set. Given a permutation σ, a reduced word for σ is a product $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ where $\ell = \ell(\sigma)$. In that problem, we showed that every permutation σ has a reduced word.

In this problem, we will prove the following result, known as Tits' Lemma: Let $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and $s_{j_1} s_{j_2} \cdots s_{j_\ell}$ be two reduced words for σ. Then we can transform $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ into $s_{j_1} s_{j_2} \cdots s_{j_\ell}$ by a sequence of the following operations.

- Replacing $s_i s_j$ by $s_j s_i$, for $|i - j| \geq 2$ and
- Replacing $s_i s_{i+1} s_i$ by $s_{i+1} s_i s_{i+1}$, or vice versa.

(1) Show that w has a reduced word ending with s_i if and only if $w(i) > w(i+1)$.
(2) Let i and j be indices between 1 and $n-1$ with $|i - j| \geq 2$. Show that, if w has a reduced word ending with s_i, and w has a reduced word ending with s_j, then w has a reduced word ending with $s_i s_j$.
(3) Let $1 \leq i < n - 1$. Show that, if w has a reduced word ending with s_i, and w has a reduced word ending with s_{i+1}, then w has a reduced word starting with $s_i s_{i+1} s_i$.
(4) Prove Tits Lemma. Hint: Induct on ℓ.