Problem Set Three: Due Friday, September 25

Problem 1. We will define the 0-Hecke monoid in class to have generators $e_1, e_2, \ldots, e_{n-1}$ and relations

$$e_i e_j = e_j e_i \quad |i-j| \geq 2$$
$$e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1}$$
$$e_i^2 = e_i$$

Let $\binom{[n]}{k}$ be the set of k-element subsets of $\{1, 2, \ldots, n\}$. Define maps $e_i : \binom{[n]}{k} \to \binom{[n]}{k}$ by

$$e_i(X) = \begin{cases} X \cup \{i+1\} \setminus \{i\} & i \in X,\ i+1 \notin X \\ X & \text{otherwise} \end{cases}.$$

Check that this is an action of the 0-Hecke monoid.

Problem 2. This problem checks some useful identities between the minors of an $m \times n$ matrix X.

(1) Let I be a k element subset of $[m]$ and let J be a $k-2$ element subset of $[n]$, with $a < b < c < d$ elements of $[n] \setminus J$. We'll abbreviate $J \cup \{a, b\}$ to Jab, and so forth. Show that

$$\Delta^I_{Jac}(X)\Delta^I_{Jbd}(X) = \Delta^I_{Jab}(X)\Delta^I_{Jcd}(X) + \Delta^I_{Jbc}(X)\Delta^I_{Jad}(X).$$

(2) Let I be a $k-1$ element subset of $[m]$ and let J be a $k-2$ element subset of $[n]$, with $a \in [m] \setminus I$ and $b < c < d$ elements of $[n] \setminus J$.

$$\Delta^{Ia}_{Jbd}(X)\Delta^I_{Jc}(X) = \Delta^{Ia}_{Jbc}(X)\Delta^I_{Jd}(X) + \Delta^{Ia}_{Jcd}(X)\Delta^I_{Jb}(X).$$

(3) Let I and J be $k-2$ element subsets of $[m]$ and $[n]$ respectively. Let $a < b$ and $c < d$ be in $[m] \setminus I$ and $[n] \setminus J$ respectively. Show that

$$\Delta^{Ia}_{Jc}(X)\Delta^{Ib}_{Jd}(X) = \Delta^I_J(X)\Delta^{Iab}_{Jcd}(X) + \Delta^{Ia}_{Jd}(X)\Delta^{Ib}_{Jc}(X).$$