PROBLEM SET FOUR: DUE FRIDAY, OCTOBER 9

Problem 1. Show the following equality of polynomials in q:

$$\sum_{q \in S_n} q^{\ell(w)} = (1+q)(1+q+q^2)(1+q+q^2+q^3)\cdots(1+q+q^2+\cdots+q^{n-1}).$$

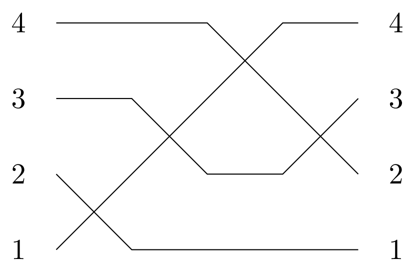
Problem 2. Consider the sequence of matrices:

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & u & uv \\ & 1 & v \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & u+w & uv \\ & 1 & v \\ & & 1 \end{bmatrix}.$$

These are the partial products of our factorization $x_1(u)x_2(v)x_3(w)$.

Factor each of these matrices into the form $(N_- w \cap w N_+) B_-$ for the appropriate permutation w.

Problem 3. This problem introduces ***wiring diagrams***, a graphical way to display words in S_n. Let $s_{i_1} s_{i_2} \cdots s_{i_r}$ be a word in S_n. A wiring diagram consists of n paths in $\mathbb{R}^2$, starting at $\{(0,y) : 1 \leq y \leq n\}$ and ending at $\{(r,y) : 1 \leq y \leq n\}$. In the region between $x = k$ and $x = k+1$, the paths at height i_k and $i_k + 1$ cross and all the others continue straight across. For example, here is a wiring diagram for the word $s_1 s_2 s_3 s_2$ in S_4:



(1) Which is the correct statement: The path starting at $(0,y)$ ends at $(0, w(y))$, or the path starting at $(0,y)$ ends at $(0, w^{-1}(y))$?

(2) Show that a word $s_{i_1} s_{i_2} \cdots s_{i_r}$ is reduced if and only if there is no pair of paths that cross more than once.

Problem 4. This problem introduces the ***strong order*** (also called Bruhat order) on the symmetric group. Given a permutation matrix w, let $r_{ab}(w)$ be the rank of the upper left $a \times b$ submatrix of w. We define $u \leq v$ in strong order iff $r_{ab}(u) \geq r_{ab}(v)$ for all (a,b).

(1) Put a partial order on the set of k element subsets of $[n]$ as follows: Let $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$; we define $I \leq J$ iff $i_a \leq j_a$ for all a. Show that $u \leq v$ in strong order if and only if $u[k] \leq v[k]$ for $1 \leq k \leq n$.

(2) Let $w \in S_n$. For any transposition $(a\ b)$ with $a < b$, show that $w(a\ b) \geq w$ if $w(a) < w(b)$ and that $w(a\ b) \leq w$ if $w(a) \geq w(b)$.

(3) Let $u < v$. Show that there is a transposition $(a\ b)$ for which $u < u(a\ b) \leq v$. I am going to leave this one without a hint, because I am curious to see how you'll do it; it is a bit harder than I would ask without one.