PROBLEM SET FIVE: DUE FRIDAY, OCTOBER 23

Problem 1. Let M be a matrix. We define a top justified consecutive minor of M to be a minor of the form $\Delta^{12\cdots}_{(a+1)(a+2)\cdots(a+k)}(M)$, for some a and k, and a left justified consecutive minor to be a minor of the form $\Delta^{(b+1)(b+2)\cdots(b+k)}_{12\cdots k}(M)$ for some b and k. We'll define a top-or-left justified minor to be a minor which is either top or left justified (or both).

(1) Show that a totally positive matrix is determined by the values of its top-or-left justified minors.

Let X be a $n \times n$ totally positive matrix. Let D be the diagonal matrix with entries

$$\left(\Delta^1_1(X), \frac{\Delta^{12}_{12}(X)}{\Delta^1_1(X)}, \frac{\Delta^{123}_{123}(X)}{\Delta^{12}_{12}(X)}, \ldots, \frac{\Delta^{[n]}[n](X)}{\Delta^{[n-1]}_{[n-1]}(X)} \right).$$

Let L be the unique lower triangular matrix such that

$$\Delta^{(b+1)(b+2)\cdots(b+k)}_{12\cdots k}(L) = \frac{\Delta^{(b+1)(b+2)\cdots(b+k)}_{12\cdots k}(X)}{\Delta^{12\cdots k}_{12\cdots k}(X)}$$

and let U be the unique upper triangular matrix such that

$$\Delta^{12\cdots k}_{(a+1)(a+2)\cdots(a+k)}(U) = \frac{\Delta^{12\cdots k}_{(a+1)(a+2)\cdots(a+k)}(X)}{\Delta^{12\cdots k}_{12\cdots k}(X)}.$$

(2) Show that $X = LDU$.

We have now confirmed the claim from earlier in the course that, if X is totally positive and has LDU factorization (L, D, U), then L, D and U are totally nonnegative.

Problem 2. The Plücker coordinates on the Grassmannian $G(2,4)$ obey the relation $\Delta^{13}\Delta^{24} = \Delta^{12}\Delta^{34} + \Delta^{14}\Delta^{23}$. A point of $G(2,4)$ is called totally nonnegative if all of its Plücker coordinates are nonnegative.

(1) Which of the subsets of $\{12, 13, 14, 23, 24, 34\}$ are capable of being the sets $\{I : \Delta^I(x) \neq 0\}$ for x a totally nonnegative point of $G(2,4)$? You should find that there are 33 in total.
(2) Give examples of points of $G(2,4)$ where each of the following occurs

$$\Delta^{12}, \Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24}, \Delta^{34} > 0$$
$$\Delta^{12}, \Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24} > 0, \quad \Delta^{34} = 0$$
$$\Delta^{13}, \Delta^{14}, \Delta^{23}, \Delta^{24} > 0, \quad \Delta^{12} = \Delta^{34} = 0$$
$$\Delta^{12}, \Delta^{13}, \Delta^{14} > 0 \quad \Delta^{23} = \Delta^{24} = \Delta^{34} = 0$$
$$\Delta^{12}, \Delta^{13}, \Delta^{23} > 0 \quad \Delta^{14} = \Delta^{24} = \Delta^{34} = 0$$