PROBLEM SET SEVEN: DUE FRIDAY, NOVEMBER 13

Problem 1. We return to the topic of the strong order on S_n. Given a permutation matrix w, let $r_{ab}(w)$ be the rank of the upper left $a \times b$ submatrix of w. We define $u \leq v$ in strong order iff $r_{ab}(u) \geq r_{ab}(v)$ for all (a, b).

We showed before that, for any permutation w and any transposition $(i\ j)$, we have $w(i\ j) > w$ if $w(i) < w(j)$ and $w(i\ j) < w$ if $w(i) > w(j)$. We want to show that, if $u < w$, then there is a transposition $(i\ j)$ with $u < u(i\ j) < w$. Thus, any two elements of S_n are linked by a chain of transpositions in strong order.

Let i be the least index for which $u(i) \neq w(i)$.

(1) Show that $u(i) < w(i)$.

Let j be the least index for which $u(i) < u(j) \leq w(i)$.

(2) Show that $u < u(i\ j) \leq w$.

Problem 2. Show that $u \leq w$, in strong order, if and only if $B_- u B_+ \cap B_+ w B_+$ is nonempty.

The next problem uses the notion of a ***Kasteleyn labelling***, which we will discuss in class. Here is the definition for those who want to start earlier: Let G be a bipartite planar graph embedded in the plane, and all of whose bounded faces are discs. A Kasteleyn labelling of G is an assignment of a nonzero complex number $\alpha(e)$ to each edge e of G with the following property: Consider any face of G, with edges e_1, e_2, …, e_{2k}. Then we impose that

$$\alpha(e_1)\alpha(e_3)\alpha(e_5)\cdots\alpha(e_{2k-1}) = (-1)^{k-1}\alpha(e_2)\alpha(e_4)\alpha(e_6)\cdots\alpha(e_{2k}).$$

Problem 3. There is a cute geometric way to find Kasteleyn labellings. Embed G in the complex plane such that, for every bounded face F of G, the vertices of F lie on a circle. For an edge e with white endpoint w and black endpoint b, let $\alpha(e)$ be the unit complex number in direction $w - b$. Prove that this is a Kasteleyn labelling.