MATH 668 PROBLEM SET 1: DUE WEDNESDAY, SEPTEMBER 7

See the course website for homework policy.

Problem 1. Use the mathematical software package of your choice to do these computations:

(1) Write down the 6 transition matrices relating the e, h and m bases for the degree 4 symmetric polynomials. (These should be 5×5 matrices.)

(2) Expand $(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)$ in the e-basis of Λ_4.

Problem 2. Let $g \in \mathrm{GL}_n(\mathbb{C})$, and let the characteristic polynomial of g factor as $\prod_{i=1}^n (x-\lambda_i)$. Let t be the diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\chi : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$ be an continuous function which is constant on conjugacy classes. Show that $\chi(g) = \chi(t)$.

Problem 3. Let G be a group acting on a complex vector space V by a map $\rho : G \to \mathrm{GL}(V)$. Recall that we define the character χ_V of this action by $\chi_V(g) = \mathrm{Tr}\rho(g)$.

(1) Let V and W be representations of G. Define the "obvious" representation of G on $V \oplus W$ and show that $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$.

(2) Let V and W be representations of G. Define the "obvious" representation of G on $V \otimes W$ and show that $\chi_{V \otimes W}(g) = \chi_V(g) \times \chi_W(g)$.

For the next several problems, let G be $\mathrm{GL}_3(\mathbb{C})$ and let $g = \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}$. Let V be the standard representation of G on $\mathbb{C}^3$.

(3) Define the "obvious" representation of G on $\bigwedge^2 V$ and compute $\chi_{\bigwedge^2 V}(g)$.

(4) Define the "obvious" representation of G on $\mathrm{Sym}^2 V$ and compute $\chi_{\mathrm{Sym}^2 V}(g)$.

(5) Define the "obvious" representation of G on $V^\vee$ and compute $\chi_{V^\vee}(g)$.

Problem 4. Let λ and μ be two partitions with $|\lambda| = |\mu|$. Show that $\lambda \preceq \mu$ if and only if $\mu^T \preceq \lambda^T$.

Problem 5. We recall from class the notations e_k and h_k:

$$e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad h_k = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In class we showed that each could be written as a polynomial in the others but didn't give an explicit formula. Remedy this by showing that

$$h_k = \det \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_k \\ 1 & e_1 & e_2 & \cdots & e_{k-1} \\ 0 & 1 & e_1 & \cdots & e_{k-2} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & e_1 \end{bmatrix} \qquad e_k = \det \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_k \\ 1 & h_1 & h_2 & \cdots & h_{k-1} \\ 0 & 1 & h_1 & \cdots & h_{k-2} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & h_1 \end{bmatrix}.$$

Problem 6. This problem will work you through the basic properties of the power symmetric functions; they are important but won't come up very often for us.

Define $p_k(x) = \sum x_i^k$ and define $p_\lambda(x) = \prod_i p_{\lambda_i}(x)$.

(1) Prove Newton's Identity:

$$k e_k = e_{k-1} p_1 - e_{k-2} p_2 + e_{k-3} p_3 - \cdots \pm e_1 p_{k-1} \mp p_k.$$

(2) Show that $\Lambda \otimes \mathbb{Q}$ is $\mathbb{Q}[p_1, p_2, p_3, \ldots]$.