MATH 668 PROBLEM SET 2: DUE WEDNESDAY, SEPTEMBER 14

See the course website for homework policy.

Problem 1. We have shown that $\Lambda = \mathbb{Z}[h_1, h_2, h_3, \ldots] = \mathbb{Z}[e_1, e_2, \ldots]$. Define $\omega : \Lambda \to \Lambda$ to be the unique ring homomorphism such that $\omega(h_n) = e_n$. Show that $\omega(e_n) = h_n$ and thus ω^2 is the identity map.

Problem 2. Show that $\bigoplus_{\ell(\lambda) \leq n} \mathbb{Z} h_\lambda$ is **not** a subring of Λ.

Problem 3. (1) Show that

$$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda e_\lambda(x) m_\lambda(y).$$

(2) Define $b_{\lambda\mu} = \langle e_\lambda, h_\mu \rangle$. Show that $b_{\lambda\mu} = b_{\mu\lambda}$.
(3) Show that the involution ω, from Problem 1, preserves $\langle \, , \, \rangle$.
(4) In class, we gave an interpretation of $\langle h_\lambda, h_\mu \rangle$ in terms of matrices of nonnegative integers with given row and column sum. Give a similar interpretation of $b_{\lambda\mu}$.

Problem 4. This problem follows up on our previous problem on power sum symmetric functions, and you may use results from that problem.

(1) Establish a formula of the form

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda z_\lambda p_\lambda(x) p_\lambda(y).$$

for some positive rational numbers z_λ. Hint: $\frac{1}{1-w} = \exp(w + w^2/2 + w^3/3 + w^4/4 + \cdots)$.
(2) Compute $\langle p_\lambda, p_\mu \rangle$ and deduce that $\langle \, , \, \rangle$ is positive definite.

Problem 5. Let 1^k be the partition $11 \cdots 1$ with k ones.

(1) Show that $\langle h_{1^k}, h_{1^k} \rangle = k!$.

Recall that, for G a group and V and W two representations of G, the space $\mathrm{Hom}^G(V, W)$ is the space of linear maps $f : V \to W$ such that $f(g \cdot \vec{v}) = g \cdot f(\vec{v})$ for all $g \in G$ and $\vec{v} \in V$. I promised you $\langle \, , \, \rangle$ would compute the dimension of the corresponding Hom space. In this case, my promise is that we should have $\dim \mathrm{Hom}^{\mathrm{GL}_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k}) = k!$, at least for k large.

(2) Give $k!$ elements of $\mathrm{Hom}^{\mathrm{GL}_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k})$ and show that these elements are linearly independent if $n > k$. (Hint: If you choose the elements I expect you to, consider what they do to $e_1 \otimes e_2 \otimes \cdots \otimes e_k$, where $e_1, e_2, \ldots, e_n$ is a basis of $\mathbb{C}^n$.)
(3) If $n < k$, give a linear dependence between your $k!$ elements of $\mathrm{Hom}^{\mathrm{GL}_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k})$.