MATH 668 PROBLEM SET 3: DUE WEDNESDAY, SEPTEMBER 21

See the course website for homework policy.

Problem 1. Give the six 5×5 matrices which convert between the Schur polynomials s_λ of degree 4, and our other favorite bases m_λ, e_λ, h_λ. (Using technology is encouraged.)

Let λ be a partition and let n be a positive integer. Let $\mathrm{SSYT}(\lambda, n)$ be the set of semistandard young Tableaux of shape λ with entries from $\{1, 2, \ldots, n\}$. In the next few problems, we will give other objects which are in bijection with $\mathrm{SSYT}(\lambda, n)$.

Problem 2. List the eight elements of $\mathrm{SSYT}((2,1),3)$.

Problem 3. A Gelfand-Tsetlin pattern is an array of $\binom{n+1}{2}$ nonnegative integers $g_i j$, for $1 \leq j \leq i \leq n$, obeying the inequalities

$$g_{ij} \geq g_{(i-1)j} \geq g_{i(j+1)}.$$

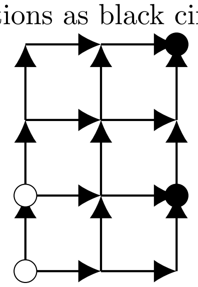
We will display them in a triangular array:

$$\begin{bmatrix} g_{11} & & & & \\ g_{21} & g_{22} & & & \\ g_{31} & g_{32} & g_{33} & & \\ \vdots & & \ddots & & \\ g_{n1} & g_{n2} & g_{n3} & \cdots & g_{nn} \end{bmatrix}.$$

(1) List the eight Gelfand-Tsetlin patterns with $(g_{31}, g_{32}, g_{33}) = (2, 1, 0)$.
(2) Give a bijection between $\mathrm{SSYT}(\lambda, n)$ and $n \times n$ Gelfand-Tsetlin patterns with $g_{nj} = \lambda_j$.

Problem 4. Define a directed graph whose vertices are the integer lattice points (i, j), and where we have edges $(i, j) \longrightarrow (i+1, j)$ and $(i, j) \longrightarrow (i, j+1)$. A ***directed path*** is a path through this graph which follows the directions of the arrows. A ***collection of vertex disjoint directed paths*** is a collection of directed paths so that no two have any vertices in common.

(1) List the eight pairs of vertex disjoint paths γ_1, γ_2, where γ_1 goes from $(1, -1)$ to $(3, 1)$ and γ_2 goes from $(1, -2)$ to $(3, -1)$. In the diagram below, the starting locations are shown as white circles and the ending locations as black circles.



(2) Give a bijection between SSYT and n-tuples of vertex disjoint paths $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ where γ_i connects $(1, -i)$ to $(n, \lambda_i - i)$.

Problem 5. Let $\epsilon : S_n \to \{\pm 1\}$ be the map which sends a permutation to its sign. A polynomial $f(x_1, x_2, \ldots, x_n)$ in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ is called ***alternating*** if

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = \epsilon(\sigma) f(x_1, x_2, \ldots, x_n)$$

for all $\sigma \in S_n$. Put $\Delta := \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

Show that a polynomial $f(x_1, x_2, \ldots, x_n)$ in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ is alternating if and only if $f(x_1, x_2, \ldots, x_n)$ is of the form $\Delta(x_1, x_2, \ldots, x_n) g(x_1, x_2, \ldots, x_n)$ for some symmetric polynomial g.