MATH 668 PROBLEM SET 4: DUE WEDNESDAY, SEPTEMBER 28

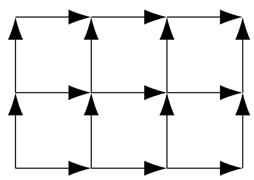
Problem 1. Let λ be a partition and put $\delta(\lambda, d) = s_\lambda(1, 1, 1, \ldots, 1)$ where there are d ones.

(1) Compute $\delta(2, d)$, $\delta((1, 1), d)$, $\delta((2, 1), d)$ and $\delta((2, 2), d)$.
(2) Show that $\delta(\lambda, d)$ is a polynomial in d.

Problem 2. The goal of this problem is to get you to fill in the details of the proof of the dual Jacobi-Trudi identity, which asserts that

$$s_{\lambda^T} = \det \begin{bmatrix} e_{\lambda_1} & e_{\lambda_1+1} & e_{\lambda_1+2} & \cdots & e_{\lambda_1+\ell} \\ e_{\lambda_2-1} & e_{\lambda_2} & e_{\lambda_2+1} & \cdots & e_{\lambda_2+\ell-1} \\ e_{\lambda_3-2} & e_{\lambda_3-1} & e_{\lambda_3} & \cdots & e_{\lambda_3+\ell-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{\lambda_\ell-\ell+1} & e_{\lambda_\ell-\ell+2} & e_{\lambda_\ell-\ell+3} & \cdots & e_{\lambda_\ell} \end{bmatrix}.$$

(1) Take a $k \times (n - k)$ rectangular grid and direct all the edges up and right. The figure below depicts $k = 2$, $n = 5$.



Show how to put weights on the edges of this graph such that the weighted sum of all paths from the lower left to the upper right is $e_k(x_1, x_2, \ldots, x_n)$.

(2) Prove the dual Jacobi-Trudi identity.

Problem 3. The point of this problem is to give a direct proof of the dual Cauchy identity:

$$\prod_{i,j=1}^{n} (1 + x_i y_j) = \sum_\lambda s_\lambda(x_1, x_2, \ldots, x_n) s_{\lambda^T}(y_1, y_2, \ldots, y_n).$$

Define $\alpha \to \beta$ to mean $\alpha^T \uparrow \beta^T$, and $\alpha \leftarrow \beta$ to mean $\alpha^T \downarrow \beta^T$.

(1) Show that, for two partitions α and β, we have $\alpha \to \beta$ iff $\alpha_j \leq \beta \leq \alpha_j + 1$ for all j.
(2) Fix partitions β and γ. Show that

$$\sum_{\beta \uparrow \delta \leftarrow \gamma} x^{|\delta|-|\beta|} y^{|\delta|-|\gamma|} = (1 + xy) \sum_{\beta \leftarrow \alpha \uparrow \gamma} y^{|\beta|-|\alpha|} x^{|\gamma|-|\alpha|}.$$

(3) Prove the dual Cauchy identity.

Problem 4. Problem 4 is removed because of flawed wording in a boundary case. It will return next week.