MATH 668 PROBLEM SET 4: DUE FRIDAY, OCTOBER 7 BECAUSE OF YOM KIPPUR

Problem 1. The goal of this problem is to give a representation theoretic meaning to the power sum symmetric functions. Let $V = \mathbb{C}^n$. Let $\mathrm{GL}_n \times S_d$ act on $V^{\otimes d}$ by:

$$(g, \sigma) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_d) = (g v_{\sigma^{-1}(1)}) \otimes (g v_{\sigma^{-1}(2)}) \otimes \cdots \otimes (g v_{\sigma^{-1}(d)}).$$

(The inverses are just to make this a left action, and aren't important for this problem.) Let σ be a permutation in S_d with cycle lengths $\lambda_1, \lambda_2, \ldots, \lambda_\ell$. Let g be a diagonalizable matrix with eigenvalues $x_1, x_2, \ldots, x_n$. Show that the trace of (g, σ), for this action, is $p_\lambda(x_1, x_2, \ldots, x_n)$.

Problem 2. As on Problem Set 3, define $\alpha \to \beta$ to mean that β is obtained from α by adding a vertical strip.

(1) Prove the dual Pieri formula. I recommend using the ratio of alternants formula, but multiple proofs exist.

$$e_k s_\lambda = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + k}} s_\mu.$$

(2) Prove the Pieri formula:

$$h_k s_\lambda = \sum_{\substack{\lambda \uparrow \mu \\ |\mu| = |\lambda| + k}} s_\mu.$$

Problem 3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. We define a ***special rim hook tableau*** to be a labeling of the boxes of λ with the colors $1, 2, \ldots, \ell$ such that

(1) In each row and column, the numbers increase weakly from left to right and top to bottom.
(2) The set of boxes of each color is either connected or empty, and contains no 2×2 boxes.
(3) If the color k is used, then the lower-left-most box colored k is in column 1 and row k.

For example, here are the 6 special rim hook tableau of shape $(6, 5, 3)$:

1	1	1	1	1	1
2	2	2	2	2	
3	3	3			

1	1	1	1	2	2
2	2	2	2	2	
3	3	3			

1	1	1	1	1	1
2	2	3	3	3	
3	3	3			

1	2	2	2	2	2
2	2	3	3	3	
3	3	3			

1	1	1	1	3	3
2	2	3	3	3	
3	3	3			

1	2	2	2	3	3
2	2	3	3	3	
3	3	3			

And here are the 6 special rim hook tableau of shape $(2, 2, 2)$:

1	1
2	2
3	3

1	2
2	2
3	3

1	1
2	3
3	3

2	2
2	3
3	3

1	3
2	3
3	3

2	3
2	3
3	3

Let $c_j(T)$ be the number of j's in T, and let $h(T)$ be the number of places where T has two vertically stacked boxes with the same entry. Show that:

$$s_\lambda = \sum_T (-1)^{h(T)} h_{c_1(T)} h_{c_2(T)} \cdots h_{c_\ell(T)}.$$

Turn over for one more problem!

Problem 4. Let A be an $n \times n$ matrix of nonnegative integers. There are $\binom{2n-2}{n-1}$ paths through this matrix from A_{11} to A_{nn} where each step either moves from A_{ij} to $A_{i(j+1)}$ or from A_{ij} to $A_{(i+1)j}$. For example, the diagram below shows the 6 paths in the case $n = 3$:

$$
\begin{bmatrix} A_{11} & \to & A_{12} & \to & A_{13} \\ & & & & \downarrow \\ A_{21} & & A_{22} & & A_{23} \\ & & & & \downarrow \\ A_{31} & & A_{32} & & A_{33} \end{bmatrix}
\quad
\begin{bmatrix} A_{11} & \to & A_{12} & & A_{13} \\ & & \downarrow & & \\ A_{21} & & A_{22} & \to & A_{23} \\ & & & & \downarrow \\ A_{31} & & A_{32} & & A_{33} \end{bmatrix}
\quad
\begin{bmatrix} A_{11} & \to & A_{12} & & A_{13} \\ & & \downarrow & & \\ A_{21} & & A_{22} & & A_{23} \\ & & \downarrow & & \\ A_{31} & & A_{32} & \to & A_{33} \end{bmatrix}
$$

$$
\begin{bmatrix} A_{11} & & A_{12} & & A_{13} \\ \downarrow & & & & \\ A_{21} & \to & A_{22} & \to & A_{23} \\ & & & & \downarrow \\ A_{31} & & A_{32} & & A_{33} \end{bmatrix}
\quad
\begin{bmatrix} A_{11} & & A_{12} & & A_{13} \\ \downarrow & & & & \\ A_{21} & \to & A_{22} & & A_{23} \\ & & \downarrow & & \\ A_{31} & & A_{32} & \to & A_{33} \end{bmatrix}
\quad
\begin{bmatrix} A_{11} & & A_{12} & & A_{13} \\ \downarrow & & & & \\ A_{21} & & A_{22} & & A_{23} \\ \downarrow & & & & \\ A_{31} & \to & A_{32} & \to & A_{33} \end{bmatrix}
$$

We define $L(A)$ to be the maximum of $\max_\gamma \left(\sum_{(i,j)\in\gamma} A_{ij} \right)$ where the maximum runs over these $\binom{2n-2}{n-1}$ paths. For example, in the $n = 3$ case,

$$
\begin{aligned}
L(A) = \max(&A_{11}+A_{12}+A_{13}+A_{23}+A_{33},\ A_{11}+A_{12}+A_{22}+A_{23}+A_{33},\ A_{11}+A_{12}+A_{22}+A_{32}+A_{33}, \\
&A_{11}+A_{21}+A_{22}+A_{23}+A_{33},\ A_{11}+A_{21}+A_{22}+A_{32}+A_{33},\ A_{11}+A_{21}+A_{31}+A_{32}+A_{33}).
\end{aligned}
$$

(1) Use the RSK bijection to turn A into a pair (T, U) of SSYTs of shape λ. Show that $\lambda_1 = L(A)$.

(2) If A is a permutation matrix, meaning that there is a permutation w such that $A_{w(j)j} = 1$ and $A_{ij} = 0$ for all other (i, j), what is the relation between $L(A)$ and the longest increasing subsequence of w?