

**MATH 668 PROBLEM SET 4:
DUE FRIDAY, OCTOBER 7 BECAUSE OF YOM KIPPUR**

Problem 1. The goal of this problem is to give a representation theoretic meaning to the power sum symmetric functions. Let $V = \mathbb{C}^n$. Let $GL_n \times S_d$ act on $V^{\otimes d}$ by:

$$(g, \sigma) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_d) = (gv_{\sigma^{-1}(1)}) \otimes (gv_{\sigma^{-1}(2)}) \otimes \cdots \otimes (gv_{\sigma^{-1}(d)}).$$

(The inverses are just to make this a left action, and aren't important for this problem.) Let σ be a permutation in S_d with cycle lengths $\lambda_1, \lambda_2, \dots, \lambda_\ell$. Let g be a diagonalizable matrix with eigenvalues x_1, x_2, \dots, x_n . Show that the trace of (g, σ) , for this action, is $p_\lambda(x_1, x_2, \dots, x_n)$.

Problem 2. As on Problem Set 3, define $\alpha \rightarrow \beta$ to mean that β is obtained from α by adding a vertical strip.

- (1) Prove the dual Pieri formula. I recommend using the ratio of alternants formula, but multiple proofs exist.

$$e_k s_\lambda = \sum_{\substack{\lambda \rightarrow \mu \\ |\mu| = |\lambda| + k}} s_\mu.$$

- (2) Prove the Pieri formula:

$$h_k s_\lambda = \sum_{\substack{\lambda \uparrow \mu \\ |\mu| = |\lambda| + k}} s_\mu.$$

Problem 3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition. We define a *special rim hook tableau* to be a labeling of the boxes of λ with the colors 1, 2, \dots , ℓ such that

- (1) In each row and column, the numbers increase weakly from left to right and top to bottom.
- (2) The set of boxes of each color is either connected or empty, and contains no 2×2 boxes.
- (3) If the color k is used, then the lower-left-most box colored k is in column 1 and row k .

For example, here are the 6 special rim hook tableau of shape $(6, 5, 3)$:

1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	2	2	2	2	2	1	1	1	1	3	3	1	2	2	2	3	3
2	2	2	2	2	2	2	2	2	2	2	2	2	2	3	3	3	2	2	3	3	3	3	2	2	3	3	3	3	2	2	3	3	3	3	
3	3	3				3	3	3				3	3	3				3	3	3				3	3	3				3	3	3			

And here are the 6 special rim hook tableau of shape $(2, 2, 2)$:

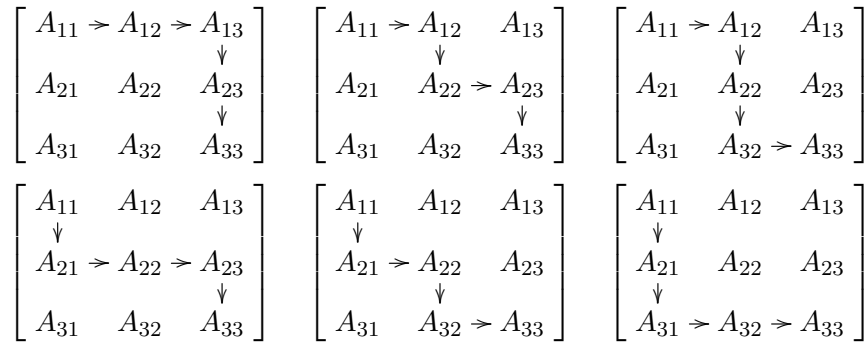
1	1	1	2	1	1	2	2	1	3	2	3
2	2	2	2	2	3	2	3	2	3	2	3
3	3	3	3	3	3	3	3	3	3	3	3

Let $c_j(T)$ be the number of j 's in T , and let $h(T)$ be the number of places where T has two vertically stacked boxes with the same entry. Show that:

$$s_\lambda = \sum_T (-1)^{h(T)} h_{c_1(T)} h_{c_2(T)} \cdots h_{c_\ell(T)}.$$

Turn over for one more problem!

Problem 4. Let A be an $n \times n$ matrix of nonnegative integers. There are $\binom{2n-2}{n-1}$ paths through this matrix from A_{11} to A_{nn} where each step either moves from A_{ij} to $A_{i(j+1)}$ or from A_{ij} to $A_{(i+1)j}$. For example, the diagram below shows the 6 paths in the case $n = 3$:



We define $L(A)$ to be the maximum of $\max_{\gamma} \left(\sum_{(i,j) \in \gamma} A_{ij} \right)$ where the maximum runs over these $\binom{2n-2}{n-1}$ paths. For example, in the $n = 3$ case,

$$L(A) = \max(A_{11} + A_{12} + A_{13} + A_{23} + A_{33}, A_{11} + A_{12} + A_{22} + A_{23} + A_{33}, A_{11} + A_{12} + A_{22} + A_{32} + A_{33}, A_{11} + A_{21} + A_{22} + A_{23} + A_{33}, A_{11} + A_{21} + A_{22} + A_{32} + A_{33}, A_{11} + A_{21} + A_{31} + A_{32} + A_{33}).$$

- (1) Use the RSK bijection to turn A into a pair (T, U) of SSYTs of shape λ . Show that $\lambda_1 = L(A)$.
- (2) If A is a permutation matrix, meaning that there is a permutation w such that $A_{w(j)j} = 1$ and $A_{ij} = 0$ for all other (i, j) , what is the relation between $L(A)$ and the longest increasing subsequence of w ?