MATH 668 PROBLEM SET 7:
DUE MONDAY, OCTOBER 31

Problem 1. The point of this exercise is to work through the representation theory of the symmetric S_3 and see that all of the claims we have made in class are true. I'll start by telling you the simple representations of S_3; there are three of them. One of them (called the trivial) is a one-dimensional representation with $\rho(g) = 1$ for all g. One of them (called the sign) is a one dimensional representation with $\rho(g) = \mathrm{sign}(g) \in \{1, -1\}$. And one of them is the two dimensional representation of S_3 on $\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]^{\perp} \subset \mathbb{R}^3$.

(1) Describe the conjugacy classes of S_3. Check that there are also three of them.
(2) Describe the characters of S_3. Check that they give an orthogonal basis for conjugacy invariant functions on S_3.
(3) Write out matrices $\rho(g)$ for the actions on each of these representations, so that the matrices are orthogonal. Check that the functions $g \mapsto \sqrt{\dim V}\rho_V(g)_{ij}$ give an orthonormal basis for functions on S_3.

Problem 2. The group $O(2)$ is the group of 2×2 real orthogonal matrices. Every matrix in $O(2)$ is either of the form $\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right]$. In this problem, we will work out the continuous representation theory of $O(2)$ in $\mathrm{GL}_n(\mathbb{C})$.

The subgroup of matrices in $O(2)$ with determinant 1 is called $SO(2)$. (In other words, $SO(2)$ is the group of rotations.) We showed in class that every continuous representation of $SO(2)$ is a direct sum of representations of the form $\rho_k : \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right] \mapsto [(\cos\theta + i \sin\theta)^k]$ for $k \in \mathbb{Z}$.

Let V be a continuous complex representation of $O(2)$. Write $V = \bigoplus_k V_k$ where $SO(2)$ acts V_k by ρ_k.

(1) Let σ be a matrix in $O(2)$ with determinant -1 (in other words, a reflection). Show that σ maps V_k to V_{-k}.
(2) Show that V_0 is a $O(2)$-subrepresentation of V and, for $k > 0$, show that $V_k \oplus V_{-k}$ is an $O(2)$-subrepresentation of V.
(3) Show that V_0 is a direct sum of copies the following two representations: The 1-dimensional trivial representation given by $\rho(g) = 1$ for all $g \in O(2)$, and the 1-dimensional representation where $\rho(g) = \det(g)$.
(4) Show that $V_k \oplus V_{-k}$ is a direct sum of copies of the 2-dimensional representation given by

$$\begin{aligned} \rho\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) &= \begin{bmatrix} (\cos\theta + i\sin\theta)^k & 0 \\ 0 & (\cos\theta + i\sin\theta)^{-k} \end{bmatrix} \\ \rho\left(\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\right) &= \begin{bmatrix} 0 & (\cos\theta + i\sin\theta)^{-k} \\ (\cos\theta + i\sin\theta)^k & 0 \end{bmatrix} \end{aligned}.$$

Since you have shown that an arbitrary representation is the direct sum of the 1- and 2-dimensional representations listed here, you have found all representations of $O(2)$.

Problem 3. Let κ be an infinite field of characteristic 0. Let G be the group of matrices of the form $\left[\begin{smallmatrix} t & u \\ 0 & 1 \end{smallmatrix}\right]$, with $t \in \kappa^{\times}$ and $u \in \kappa$. (As an abstract group, $G \cong \kappa^{\times} \ltimes \kappa^{+}$.) Let $\rho : G \to \mathrm{GL}(V)$ be an algebraic representation, meaning representations whose matrix entries are in $k[t, t^{-1}, u]$.

You have already shown, and may assume:

- That V decomposes as $V = \bigoplus V_k$, so that $\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]$ acts on V_k by t^k
- That there is a nilpotent matrix N such that $\rho\left(\left[\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right]\right) = \sum_{k=0}^{\infty} \frac{u^k N^k}{k!}$

(1) Show that N maps V_k to V_{k+1}.
(2) Given a finite dimensional vector space V with a direct sum decomposition $V = \bigoplus V_k$, and a nilpotent operator N mapping V_k to V_{k+1}, show that there is a corresponding algebraic representation of G.