MATH 668 PROBLEM SET 8:
DUE ?????

Problem 1.
(1) Write down bases for $\bigwedge^2 \mathbb{C}^3$ and $\mathrm{Sym}^2\mathbb{C}^2$.
(2) Write down the matrix for the natural action of $g = \left[\begin{smallmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{smallmatrix}\right]$ on $\bigwedge^2 \mathbb{C}^3$.
(3) Write down the matrix for the natural action of $g = \left[\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right]$ on $\mathrm{Sym}^2\mathbb{C}^2$.

Problem 2. This problem works through the main results about Schur-Weyl duality, which concerns the relationship between $\mathrm{GL}_n(\mathbb{C})$ and S_n representations. Let $T = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}$ be the torus in GL_n. If V is any algebraic representation of GL_n, then let $V_{1\cdots 1}$ be the subspace on which T acts by the character $(1, 1, \dots, 1)$. (This may be 0-dimensional.)

(1) Show that $S_n \subset \mathrm{GL}_n$ maps V_1 to itself.
(2) Let λ be a partition and let V_λ be the corresponding polynomial representation of V_λ. Show that $(V_\lambda)_{1\cdots 1}$ is nonzero if and only if $|\lambda| = n$. For λ a partition of n, define $W_\lambda = (V_\lambda)_{1\cdots 1}$.

Let R be the n^2-dimensional polynomial ring $\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}]$. I'll write ${}_{1\cdots 1}R_{1\cdots 1}$ for the subspace of R where T acts by character $(1, 1, \dots, 1)$ for both the left and right action of T, and I'll write $R_{1\cdots 1}$ for the subspace where T acts by $(1, \dots, 1)$ from the right.

(3) Show that ${}_{1\cdots 1}R_{1\cdots 1} \cong \mathbb{C}S_n$ as an $S_n \times S_n$ representation. Show that $R_{1\cdots 1} \cong (\mathbb{C}^n)^{\otimes n}$ as a $\mathrm{GL}_n \times S_n$ represenation.
(4) Show that ${}_{1\cdots 1}R_{1\cdots 1} \cong \bigoplus_{|\lambda|=n} W_\lambda \otimes W_\lambda^\vee$ as an $S_n \times S_n$ representation. Show that $R_{1\cdots 1} \cong \bigoplus_{|\lambda|=n} V_\lambda \otimes W_\lambda^\vee$ as a $\mathrm{GL}_n \times S_n$ representation.
(5) Using the identity $\mathbb{C}S_n \cong {}_{1\cdots 1}R_{1\cdots 1} \cong \bigoplus_{|\lambda|=n} W_\lambda \otimes W_\lambda^\vee$, show that the W_λ are a complete list of the irreducible representations of S_n.

Problem 3. This problem studies the Young symmetrizer. Let λ be a partition of n. Choose a numbering of the boxes of λ by $[n]$. Let $S_R \subset S_n$ be the subgroup which permutes the boxes within rows, so $S_R \cong \prod S_{\lambda_k}$; let S_C be the subgroup which permutes the boxes within columns, so $S_C \cong \prod S_{\lambda_k^T}$. We begin with some pure combinatorics:

(1) Let π be a permutation in S_n. Show that exactly one of the following is true: Either π factors as $\rho\sigma$ for $\rho \in S_R$ and $\sigma \in S_C$, or else there are two boxes i and j in the same column such that $\pi(i)$ and $\pi(j)$ are in the same row.

We define $\alpha_\lambda = \sum_{\sigma \in S_C} (-1)^\sigma \sigma$ and $\beta_\lambda = \sum_{\rho \in S_R} \rho$, as elements of the group algebra.

(2) Show that, for any permutation $\pi \in S_n$, we either have $\beta_\lambda \pi \alpha_\lambda = \pm \beta_\lambda \alpha_\lambda$, or else $\beta_\lambda \pi \alpha_\lambda = 0$.
(3) The Young symmetrizer is defined as $\omega_\lambda := \beta_\lambda \alpha_\lambda$. Show that ω_λ^2 is a scalar multiple of ω_λ.