MATH 668 PROBLEM SET 9: DUE FRIDAY, NOVEMBER 18 LAST PROBLEM SET!

Problem 1. Let V_{311} be the irreducible representation of GL_3 with character $s_{31}(x_1, x_2, x_3)$. We'll write $\rho : \mathrm{GL}_3 \to \mathrm{GL}(V_{31})$ for the representation homomorphism.

(1) Write down a basis of V_{31} in terms of your favorite construction of V_{31}. Hint: $\dim V_{31}$ is 15.

(2) Write down the action of $\rho\left(\begin{bmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$ on your basis. (It should break into many small blocks, so there isn't as much to write as you might fear.

(3) Let $r : \mathrm{Mat}_{3\times 3} \to \mathrm{End}(V)$ be the corresponding Lie algebra map. Write down the action of $\rho\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$ on your basis.

Problem 2. Let λ be a partition of n. We first review our construction of V_λ in terms of products of matrix minors.

We work with $n \times n$ matrices whose entries are variables z_{ij}. For $J = \{j_1, j_2, \ldots, j_k\}$ a k-element subset of $[n]$, let

$$\Delta_J = \det \begin{bmatrix} z_{1j_1} & z_{1j_2} & \cdots & z_{1j_k} \\ z_{2j_1} & z_{2j_2} & \cdots & z_{2j_k} \\ \vdots & \vdots & \ddots & \vdots \\ z_{kj_1} & z_{kj_2} & \cdots & z_{kj_k} \end{bmatrix}.$$

If T is a tableau (semistandard or not) with columns J^1, J^2, ..., J^m, then we put $\Delta(T) = \prod_j \Delta_{J^j}$. We showed that the span of $\Delta(T)$, with T of shape λ, is V_λ. In this problem, we will show that the $\Delta(T)$ with T semi-standard form a basis of V_λ.

(1) To check that you understand the definitions, write down

$$\Delta\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right),\ \Delta\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right),\ \Delta\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\right),\ \Delta\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right),\ \Delta\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\right) \text{ and } \Delta\left(\begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array}\right)$$

explicitly as polynomials in the z_{ij}.

(2) Write $\Delta\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\right)$ as a linear combination of $\Delta\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)$ and $\Delta\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right)$. Write $\Delta\left(\begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array}\right)$ as a linear combination of $\Delta\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right)$ and $\Delta\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\right)$.

Choose any matrix w_{ij} of positive integers such that, for $i_1 < i_2$ and $j_1 < j_2$, we always have $w_{i_1j_1} + w_{i_2j_2} > w_{i_1j_2} + w_{i_2j_1}$. An explicit example is to take $w_{ij} = ij$. Put an order on the set of monomials $\{\prod z_{ij}^{A_{ij}}\}$ by defining $\prod z_{ij}^{A_{ij}} \succ \prod z_{ij}^{B_{ij}}$ if $\sum A_{ij}w_{ij} > \sum B_{ij}w_{ij}$ (and breaking ties arbitrarilary). For a nonzero polynomial $f \in \mathbb{C}[z_{11}, z_{12}, \ldots, z_{nn}]$, we define the ***leading monomial*** of f to be the largest monomial with nonzero coefficient in f.

(3) Let T be a tableau with strictly increasing columns. Describe the leading term of $\Delta(T)$.

(4) Show that, if T and U are distinct semistandard Young tableaux, then $\Delta(T)$ and $\Delta(U)$ have different terms. Conclude that $\{\Delta(T) : T \in \mathrm{SSYT}(\lambda)\}$ is a basis for V_λ.