Problem 1. Let $V_{311}$ be the irreducible representation of $GL_3$ with character $s_{31}(x_1, x_2, x_3)$. We’ll write $\rho : GL_3 \to GL(V_{31})$ for the representation homomorphism.

1. Write down a basis of $V_{31}$ in terms of your favorite construction of $V_{31}$. Hint: $\dim V_{31}$ is 15.

2. Write down the action of $\rho \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$ on your basis. (It should break into many small blocks, so there isn’t as much to write as you might fear.

3. Let $\rho : Mat_{3 \times 3} \to \text{End}(V)$ be the corresponding Lie algebra map. Write down the action of $\rho \left( \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$ on your basis.

Problem 2. Let $\lambda$ be a partition of $n$. We first review our construction of $V_\lambda$ in terms of products of matrix minors.

We work with $n \times n$ matrices whose entries are variables $z_{ij}$. For $J = \{j_1, j_2, \ldots, j_k\}$ a $k$-element subset of $[n]$, let

$$\Delta_J = \begin{vmatrix} z_{1j_1} & z_{1j_2} & \cdots & z_{1j_k} \\ z_{2j_1} & z_{2j_2} & \cdots & z_{2j_k} \\ \vdots & \vdots & \ddots & \vdots \\ z_{kj_1} & z_{kj_2} & \cdots & z_{kj_k} \end{vmatrix}.$$ 

If $T$ is a tableau (semistandard or not) with columns $J^1, J^2, \ldots, J^m$, then we put $\Delta(T) = \prod_j \Delta_J$.

We showed that the span of $\Delta(T)$, with $T$ of shape $\lambda$, is $V_\lambda$. In this problem, we will show that the $\Delta(J)$ with $T$ semi-standard form a basis of $V_\lambda$.

1. To check that you understand the definitions, write down

$$\Delta \left( \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \right), \Delta \left( \begin{bmatrix} 2 & 1 \\ 3 \end{bmatrix} \right), \Delta \left( \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \right), \Delta \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) \text{ and } \Delta \left( \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \right)$$

explicitly as polynomials in the $z_{ij}$.

2. Write $\Delta \left( \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \right)$ as a linear combination of $\Delta \left( \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \right)$ and $\Delta \left( \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \right)$. Write $\Delta \left( \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \right)$ as a linear combination of $\Delta \left( \begin{bmatrix} 1 & 2 \\ 4 \end{bmatrix} \right)$ and $\Delta \left( \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix} \right)$.

Choose any matrix $w_{ij}$ of positive integers such that, for $i_1 < i_2$ and $j_1 < j_2$, we always have $w_{i_1j_1} + w_{i_2j_2} > w_{i_1j_2} + w_{i_2j_1}$. An explicit example is to take $w_{ij} = ij$. Put an order on the set of monomials $\{ \prod z_{ij}^{A_{ij}} \}$ by defining $\prod z_{ij}^{A_{ij}} > \prod z_{ij}^{B_{ij}}$ if $\sum A_{ij} w_{ij} > \sum B_{ij} w_{ij}$ (and breaking ties arbitrarily). For a nonzero polynomial $f \in \mathbb{C}[z_{11}, z_{12}, \ldots, z_{nn}]$, we define the leading monomial of $f$ to be the largest monomial with nonzero coefficient in $f$.

3. Let $T$ be a tableau with strictly increasing columns. Describe the leading term of $\Delta(T)$.

4. Show that, if $T$ and $U$ are distinct semistandard Young tableaux, then $\Delta(T)$ and $\Delta(U)$ have different terms. Conclude that $\{ \Delta(T) : T \in \text{SSYT}(\lambda) \}$ is a basis for $V_\lambda$. 