POTENTIAL PAPER/PRESENTATION TOPICS

This course will require a final expository paper or oral presentation on some subject in algebraic geometry. I am imagining a paper of $8-15$ pages in length and a talk of 25-50 minutes. The paper will be due Wedneday, December 7. Talk schedules will be determined once we know how many speakers there are.

I'd be glad to talk to you about your interests and how to find paper topics that might fit them. I also am glad to keep talking to you as you work on the paper.

Here are some ideas for potential topics:

Pure combinatorial topics

(1) There are "piecewise linear" models of pretty much everything we discuss in this course. We saw a bit of this when we learned about Gelfand-Tsetlin patterns, but there are also piecewise linear versions of RSK, of crystal operators, of the Littelwood-Richardson rule. Learn about some of them and explain them. Some starting sources are [1, 2, 3, 4, 5].

(2) There are interesting papers on the structure of a generic partition or a generic semistandard Young tableaux. Talk about both or either of these. Some sources are [6, 7], and the sources they cite.

(3) Greta Panova, Igor Pak, and their collaborators have done fascinating work on studying the computational complexity of combinatorial constructions such as Young tableaux, the Littlewood-Richardson rule and so forth. A useful survey paper from the early 2000's is [8]; a very exciting recent paper is [9]. I'd be glad to hear about either; probably combining them would be too much.

Algebraic geometry/commutative algebra

(4) We will describe representations of GL_n as functions on GL_n, which can be organized into a ring called the Plücker algebra. What do results on Gr"obner bases and other tools of combinatorial commutative algebra say about making bases for these representations? My usual preferred reference for this is [10, Chapter 14].

(5) One can also think about representations of GL_n as sections of line bundles on the flag variety. One can use the characteristic p technique of "Frobenius splitting" to study these; explain some of the results. The main paper I know is [11], coupled with the textbook [12]. Allen Knutson's paper [13] is also very good.

(6) The flag variety contains subvarieties known as Schubert varieties. The sections of line bundles on Schubert varieties are called "Demazure modules", and they are related to combinatorial topics known as "keys". I'm not sure what good references to point you to are here.

Other combinatorial models for representations of GL_n

(7) Kuperberg's theory of webs [14] provides a really elegant combinatorial basis for SL_3 representations, which is very different from the standard monomial basis we construct. Other

good sources are [15] and [16] . Just explaining the basic theory is a fine topic; there are also connections to cluster algebras [17] and some interesting work on SL_n [18] for $n > 3$.

Finite characteristic representation theory of GL_n

(8) Let k be an infinite, algebraically closed field of characteristic p. What does the representation theory of $\mathrm{GL}_n(k)$ over the field k look like? It will no longer be true that every representation is a direct sum of simple representations; understanding extensions between representations is a major challenge here. This is an open area, and you definitely will not be able to say everything. I am not sure what the best sources are here.

(9) Let $\mathbb{F}_p$ be the finite field with p elements. What does the representation theory of $\mathrm{GL}_n(\mathbb{F}_p)$ over $\mathbb{C}$ look like? Even $\mathrm{GL}_2(\mathbb{F}_p)$ would make a good topic. Roughly speaking, there are three types of representations, plus representations which mix them together: The "principal series" representations look roughly like GL_n-representations, the "Steinberg representations" look roughly like S_n-representations, and the "cuspidal representations" are very subtle and hard to describe. I am not sure what the best sources are here.

Other Lie types

(10) The special linear groups SL_n are one of the four infinite families of simple complex Lie groups; the others are $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$. The representation theory of these groups has most of the same structure as that of SL_n; presenting this theory would make an excellent talk or paper. I learned this material from [19]; I'm not sure if that is the best source any more.

Schubert calculus

(11) Relations between symmetric polynomials and the cohomology of the Grassmannian. There are many classical sources for this, I like Fulton's [20].

(12) The K_0-theory of the Grassmannian is also expressed in terms of Young tableaux, and gets you closer to GL_n-representation theory. Here [21] is a good source for the combinatorics, and I'm not sure what the best source is for the basics of K_0-theory. I've also never quite understood how [23] and [21] fit together; maybe you can explain it to me.

(13) Derksen, Schoefield and Weyman [22] have a very beautiful construction which bijects the number of solutions to a Schubert calculus question with a basis for invariants in a tensor product of GL_n-representations. A paper spelling this out would be really nice. I think there should also be a simpler version that turns solutions into Schubert calculus questions into bases for GL_n-representations, and it would be good to see this worked out.

Representation theory of S_n

(14) Constructing the representations of S_n in the standard way, by building Young idempotents inside $\mathbb{C}[S_n]$. Connections to Young tableaux and symmetric polynomials.

(15) The Vershik-Okounkov approach to constructing the representations of S_n [24]. Here you will see standard Young tableau emerge immediately from the structure of S_n.

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