PROBLEMS ON REPRESENTATION THEORY

Let G be a group and k a field. A ***representation*** of G is a k-vector space V and a group homomorphism $\rho : G \to \mathrm{GL}(V)$. In other words, a representation is an action of G on V by linear maps. We will often denote the representation as "V", and we will sometimes write ρ_V for ρ.

Given two representations (V_1, ρ_1) and (V_2, ρ_2), the direct sum representation is the action of G on $V_1 \oplus V_2$ where g acts by $\left[\begin{smallmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{smallmatrix}\right]$. A representation V is called ***indecomposable*** if $V \neq 0$ and we cannot write $V = V_1 \oplus V_2$ for V_1, $V_2 \neq 0$. It is easy to see that any finite dimensional representation is a direct sum of indecomposable representations.

Given a representation V, a ***subrepresentation*** of V is a vector subspace U of V such that G maps U to U. A representation V is called ***simple*** if $V \neq 0$ and the only subrepresentations of V are V and $\{0\}$.

Given two G-representations U and V, a ***morphism of G-representations*** is a linear map $\phi : U \to V$ obeying

$$\phi(g * u) = g * \phi(u) \quad \text{for } g \in G,\ u \in U.$$

Problem 1. Let U and V be G-representations and let $\phi : U \to V$ be a morphism of G-representations. We write $\mathrm{Hom}_G(U, V)$ for the space of morphism of G-representations from U to V. Show that $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$ are subrepresentations of U and V respectively.

Our first goal is to prove:

Theorem (Maschke's Theorem, first version)**.** Let k have characteristic zero and let G be a finite group. Then V is simple if and only if it is indecomposable..

Problem 2. Show that, if V is not indecomposable then V is not simple.

Problem 3. Let k have characteristic zero, let G be a finite group and let V be a finite dimensional nonzero representation of G. Suppose that V is not simple, so that $U \subset V$ is a nontrivial subrepresentation. Let $\eta : V \to U$ be a linear map with $\eta(u) = u$ for $u \in U$. Define

$$\pi(v) = \tfrac{1}{|G|} \sum_{g \in G} \left(\rho(g) \circ \eta \circ \rho(g)^{-1} \right)(v).$$

(1) Show that π is a morphism of G-representations (even though η may not be).
(2) Show that π has image U and that $\pi(u) = u$ for $u \in U$.
(3) Show that $V = U \oplus \mathrm{Ker}(\pi)$. Deduce that V is not indecomposable.

Thus we conclude:

Theorem (Maschke's Theorem, second version)**.** Let k have characteristic zero, let G be a finite group and let V be a finite dimensional representation of G. Then V is a direct sum of simple representations.

We next will consider the question of whether the decomposition of V as a direct sum of simple representations is unique.

Problem 4 (Schur's Lemma, first version)**.** Let U and V be simple representations of G and let $\phi : U \to V$ be a morphism of G-representations. Show that either $\phi = 0$ or else ϕ is an isomorphism.

Problem 5. Let U_1, U_2, …, U_k be a collection of nonisomorphic simple G-representations and let $V = \bigoplus_{j=1}^k U_j^{\oplus c_j}$ for some nonnegative integers c_j.

(1) Show that $\dim_k \mathrm{Hom}_G(U_j, V) = c_j \dim_k \mathrm{Hom}_G(U_j, U_j)$.

(2) Deduce that, if $\bigoplus_{j=1}^k U_j^{\oplus c_j} \cong \bigoplus_{j=1}^k U_j^{\oplus d_j}$, then $(c_1, c_2, \ldots, c_k) = (d_1, d_2, \ldots, d_k)$. (Nitpicker alert: Did you use that $U_j \neq 0$?)

We thus deduce:

Theorem. Let G be a group and let V be a finite dimensional representation of G which can be written as $V = \bigoplus_{j=1}^k U_j^{\oplus c_j}$ for a collection of nonisomorphic simple representations U_1, U_2, …, U_k of G. Then the U_j (up to isomorphism) and the multiplicities c_j are uniquely determined by V.

This theorem doesn't require that G is finite or that k has characteristic zero, but without those hypotheses, Maschke's theorem does not apply, so there may be many representations V which cannot be written in this way.

We now see how things are better if k is algebraically closed.

Problem 6. Let k be an algebraically closed field and let U be a simple G-representation. Let $\phi : U \to U$ be a morphism of G-representations. Show that ϕ is a scalar multiple of the identity. (Hint: Let λ be an eigenvalue of ϕ, and apply Problem 4 to the linear map $\phi - \lambda \mathrm{Id}$.)

Problem 7. Let k be an algebraically closed field. Let U_1, U_2, …, U_k be a collection of nonisomorphic simple G-representations and let $V = \bigoplus_{j=1}^k U_j^{\oplus c_j}$ for some nonnegative integers c_j. Show that $\dim_k \mathrm{Hom}_G(U_j, V) = c_j$.

We now introduce character theory into the story. Given a representation V, the ***character*** of V is the function $G \to k$ defined by

$$\chi_V(g) = \mathrm{Tr}(\rho_V(g)).$$

Problem 8. Suppose that g_1 and g_2 are conjugate elements of G, meaning that $g_2 = hg_1h^{-1}$ for some $h \in G$. For any character χ of G, show that $\chi(g_1) = \chi(g_2)$.

Problem 9. Check that $\chi_{V_1 \oplus V_2}(g) = \chi_{V_1}(g) + \chi_{V_2}(g)$.

Problem 10. Let k have characteristic zero, let G be a finite group, and let V be a finite-dimensional representation of G. Let

$$V^G = \{v \in V : g(v) = v \ \forall g \in G\}.$$

Define $\pi : V \to V$ by

$$\pi(v) = \tfrac{1}{|G|} \sum_{g \in G} g * v.$$

(1) Show that $\mathrm{Im}(\pi) = V^G$ and $V = V^G \oplus \mathrm{Ker}(\pi)$.
(2) Show that $\mathrm{Tr}(\pi) = \dim V^G$.
(3) Deduce that

$$\dim V^G = \tfrac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

Problem 11. Let k have characteristic zero, let G be a finite group, and let U and V be finite-dimensional representations of G. Let $\mathrm{Hom}(U, V)$ be the vector space of k-linear maps $U \to V$ (all maps, not just the morphisms of G-representations). Define an action of G on $\mathrm{Hom}(U, V)$ by

$$(g * \phi)(u) = \left(\rho_V(g) \circ \phi \circ \rho_U(g^{-1})\right)(u).$$

(1) Show that $\mathrm{Hom}_G(U, V) = \mathrm{Hom}(U, V)^G$ for this G-action.
(2) Show that

$$\dim_k \mathrm{Hom}_G(U, V) = \tfrac{1}{|G|} \sum_{g \in G} \chi_U(g^{-1}) \chi_V(g).$$

We can improve the formula above in the case that $k = \mathbb{C}$.

Problem 12. Let G be a finite group, and let V be a finite dimensional representation of G over the field $\mathbb{C}$. Let $g \in G$.

(1) Show that all the eigenvalues of $\rho_V(g)$ are roots of unity.
(2) Show that $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, where $\overline{z}$ is the complex conjugate of z.

Thus, let G be a finite group and let p and q be $\mathbb{C}$-valued functions on g. Define

$$\langle p, q \rangle := \tfrac{1}{|G|} \sum_{g \in G} \overline{p(g)} q(g).$$

This is a positive definite Hermitian inner product. Then our results above specialize to say:

Theorem. With the above assumptions and notations, let U and V be two representations of G over $\mathbb{C}$. Then

$$\dim \mathrm{Hom}_G(U, V) = \langle \chi_U, \ \chi_V \rangle.$$

In particular, if U is a simple representation, then $\langle \chi_U, \ \chi_V \rangle$ is the multiplicity of the summand U in V.