Let $G$ be a group and $k$ a field. A **representation** of $G$ is a $k$-vector space $V$ and a group homomorphism $\rho : G \to \text{GL}(V)$. In other words, a representation is an action of $G$ on $V$ by linear maps. We will often denote the representation as “$V$”, and we will sometimes write $\rho_V$ for $\rho$.

Given two representations $(V_1, \rho_1)$ and $(V_2, \rho_2)$, the direct sum representation is the action of $G$ on $V_1 \oplus V_2$ where $g$ acts by $\rho_1(g) \cdot \rho_2(g)$. A representation $V$ is called **indecomposable** if $V \neq 0$ and we cannot write $V = V_1 \oplus V_2$ for $V_1, V_2 \neq 0$. It is easy to see that any finite dimensional representation is a direct sum of indecomposable representations.

Given a representation $V$, a **subrepresentation** of $V$ is a vector subspace $U$ of $V$ such that $G$ maps $U$ to $U$. A representation $V$ is called **simple** if $V \neq 0$ and the only subrepresentations of $V$ are $V$ and $\{0\}$.

Given two $G$-representations $U$ and $V$, a **morphism of $G$-representations** is a linear map $\phi : U \to V$ obeying

$$\phi(g \cdot u) = g \cdot \phi(u) \quad \text{for } g \in G, \ u \in U.$$  

**Problem 1.** Let $U$ and $V$ be $G$-representations and let $\phi : U \to V$ be a morphism of $G$-representations. We write $\text{Hom}_G(U, V)$ for the space of morphism of $G$-representations from $U$ to $V$. Show that $\text{Ker}(\phi)$ and $\text{Im}(\phi)$ are subrepresentations of $U$ and $V$ respectively.

Our first goal is to prove:

**Theorem** (Maschke’s Theorem, first version). Let $k$ have characteristic zero and let $G$ be a finite group. Then $V$ is simple if and only if it is indecomposable.

**Problem 2.** Show that, if $V$ is not indecomposable then $V$ is not simple.

**Problem 3.** Let $k$ have characteristic zero, let $G$ be a finite group and let $V$ be a finite dimensional nonzero representation of $G$. Suppose that $V$ is not simple, so that $U \subset V$ is a nontrivial subrepresentation. Let $\eta : V \to U$ be a linear map with $\eta(u) = u$ for $u \in U$. Define

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ \eta \circ \rho(g)^{-1})(v).$$

(1) Show that $\pi$ is a morphism of $G$-representations (even though $\eta$ may not be).

(2) Show that $\pi$ has image $U$ and that $\pi(u) = u$ for $u \in U$.

(3) Show that $V = U \oplus \text{Ker}(\pi)$. Deduce that $V$ is not indecomposable.

Thus we conclude:

**Theorem** (Maschke’s Theorem, second version). Let $k$ have characteristic zero, let $G$ be a finite group and let $V$ be a finite dimensional representation of $G$. Then $V$ is a direct sum of simple representations.
We next will consider the question of whether the decomposition of $V$ as a direct sum of simple representations is unique.

**Problem 4** (Schur’s Lemma, first version). Let $U$ and $V$ be simple representations of $G$ and let $\phi : U \to V$ be a morphism of $G$-representations. Show that either $\phi = 0$ or else $\phi$ is an isomorphism.

**Problem 5.** Let $U_1, U_2, \ldots, U_k$ be a collection of nonisomorphic simple $G$-representations and let $V = \bigoplus_{j=1}^{k} U_j^{c_j}$ for some nonnegative integers $c_j$.

1. Show that $\text{dim}_k \text{Hom}_G(U_j, V) = c_j \text{dim}_k \text{Hom}_G(U_j, U_j)$.
2. Deduce that, if $\bigoplus_{j=1}^{k} U_j^{c_j} \cong \bigoplus_{j=1}^{k} U_j^{d_j}$, then $(c_1, c_2, \ldots, c_k) = (d_1, d_2, \ldots, d_k)$. (Nitpicker alert: Did you use that $U_j \neq 0$?)

We thus deduce:

**Theorem.** Let $G$ be a group and let $V$ be a finite dimensional representation of $G$ which can be written as $V = \bigoplus_{j=1}^{k} U_j^{c_j}$ for a collection of nonisomorphic simple representations $U_1, U_2, \ldots, U_k$ of $G$. Then the $U_j$ (up to isomorphism) and the multiplicities $c_j$ are uniquely determined by $V$.

This theorem doesn’t require that $G$ is finite or that $k$ has characteristic zero, but without those hypotheses, Maschke’s theorem does not apply, so there may be many representations $V$ which cannot be written in this way.

We now see how things are better if $k$ is algebraically closed.

**Problem 6.** Let $k$ be an algebraically closed field and let $U$ be a simple $G$-representation. Let $\phi : U \to U$ be a morphism of $G$-representations. Show that $\phi$ is a scalar multiple of the identity. (Hint: Let $\lambda$ be an eigenvalue of $\phi$, and apply Problem 4 to the linear map $\phi - \lambda \text{Id}$.)

**Problem 7.** Let $k$ be an algebraically closed field. Let $U_1, U_2, \ldots, U_k$ be a collection of nonisomorphic simple $G$-representations and let $V = \bigoplus_{j=1}^{k} U_j^{c_j}$ for some nonnegative integers $c_j$. Show that $\text{dim}_k \text{Hom}_G(U_j, V) = c_j$. 
We now introduce character theory into the story. Given a representation $V$, the character of $V$ is the function $G \to k$ defined by

$$\chi_V(g) = \text{Tr}(\rho_V(g)).$$

**Problem 8.** Suppose that $g_1$ and $g_2$ are conjugate elements of $G$, meaning that $g_2 = hg_1h^{-1}$ for some $h \in G$. For any character $\chi$ of $G$, show that $\chi(g_1) = \chi(g_2)$.

**Problem 9.** Check that $\chi_{V_1 \oplus V_2}(g) = \chi_{V_1}(g) + \chi_{V_2}(g)$.

**Problem 10.** Let $k$ have characteristic zero, let $G$ be a finite group, and let $V$ be a finite-dimensional representation of $G$. Let $V^G = \{v \in V : g(v) = v \forall g \in G\}$.

Define $\pi : V \to V$ by

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g * v.$$

(1) Show that $\text{Im}(\pi) = V^G$ and $V = V^G \oplus \text{Ker}(\pi)$.

(2) Show that $\text{Tr}(\pi) = \dim V^G$.

(3) Deduce that

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

**Problem 11.** Let $k$ have characteristic zero, let $G$ be a finite group, and let $U$ and $V$ be finite-dimensional representations of $G$. Let $\text{Hom}(U, V)$ be the vector space of $k$-linear maps $U \to V$ (all maps, not just the morphisms of $G$-representations). Define an action of $G$ on $\text{Hom}(U, V)$ by

$$(g * \phi)(u) = (\rho_V(g) \circ \phi \circ \rho_U(g^{-1})) (u).$$

(1) Show that $\text{Hom}_G(U, V) = \text{Hom}(U, V)^G$ for this $G$-action.

(2) Show that

$$\dim_k \text{Hom}_G(U, V) = \frac{1}{|G|} \sum_{g \in G} \chi_U(g^{-1}) \chi_V(g).$$

We can improve the formula above in the case that $k = \mathbb{C}$.

**Problem 12.** Let $G$ be a finite group, and let $V$ be a finite dimensional representation of $G$ over the field $\mathbb{C}$. Let $g \in G$.

(1) Show that all the eigenvalues of $\rho_V(g)$ are roots of unity.

(2) Show that $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, where $\overline{z}$ is the complex conjugate of $z$.

Thus, let $G$ be a finite group and let $p$ and $q$ be $\mathbb{C}$-valued functions on $g$. Define

$$\langle p, q \rangle := \frac{1}{|G|} \sum_{g \in G} p(g)q(g).$$

This is a positive definite Hermitian inner product. Then our results above specialize to say:

**Theorem.** With the above assumptions and notations, let $U$ and $V$ be two representations of $G$ over $\mathbb{C}$. Then

$$\dim \text{Hom}_G(U, V) = \langle \chi_U, \chi_V \rangle.$$  

In particular, if $U$ is a simple representation, then $\langle \chi_U, \chi_V \rangle$ is the multiplicity of the summand $U$ in $V$. 