PROBLEMS ON REPRESENTATION THEORY

Let G be a connected complex Lie group, let $\sigma : G \to G$ be an antiholomorphic involution and let K be the fixed points of σ. Then K is a real Lie-subgroup of G. We proved, and you may assume:

Theorem. Any holomorphic function f on G which restricts to 0 on K is 0 on G.

For a complex vector space V and a representation $\rho : G \to \mathrm{GL}(V)$, we say that V is a ***holomorphic representation*** of G if the matrix entries of ρ are holomorphic functions.

The first two problems were done in the previous class, but please talk them over. Most of these problems are very short:

Problem 1. Let V and W be holomorphic representations of G. Show that $\mathrm{Hom}_K(V, W) = \mathrm{Hom}_G(V, W)$. (I.e. a linear map $\phi : V \to W$ is a map of G-representations if and only if it is a map of K-representations.)

Problem 2. Let V and W be holomorphic representations of G. Show that $V \cong W$ as G-representations if and only if $V \cong W$ as K-representations.

Problem 3. Let W be a holomorphic G-representation and let V be a vector subspace of W. Show that V is a sub-G-representation if and only if it is a sub-K-representation.

Problem 4. Let W be a holomorphic representation of G. Show that W is simple as a G-representation if and only if W is simple as a K-representation.

Problem 5. Let W be a holomorphic representation of G. Show that W is indecomposable as a G-representation if and only if W is indecomposable as a K-representation.

Now, suppose that K is compact.

Problem 6. Let W be a holomorphic representation of G. Show that W is indecomposable as a G-representation if and only if W is simple as a G-representation.