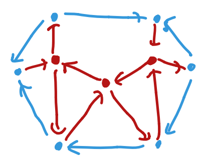
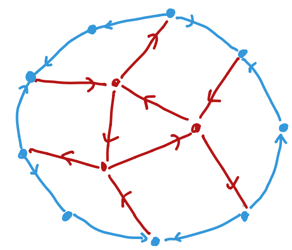
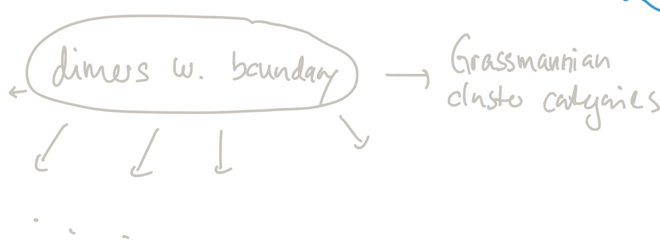
12/8/2020

Dimer models with boundary, Grassmannian cluster categories & friezes (I)



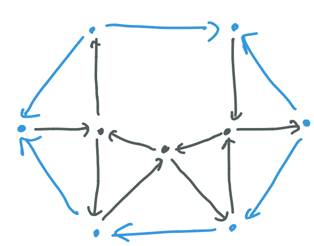




A <u>dimer model w. boundary</u> is a finite quiver Q that embeds in a surface S s.t. each connected component of $S \setminus Q$ is simply connected and bounded by an oriented cycle. These cycles are the <u>unit cycles</u>.

Arrows of Q are <u>internal</u> if they are contained in 2 faces, boundary if they belong to one face only.

The vertices incident w. boundary arrows are boundary vertices.



Examples   Quivers arising from

Q

① * triangulations of disks; of surfaces

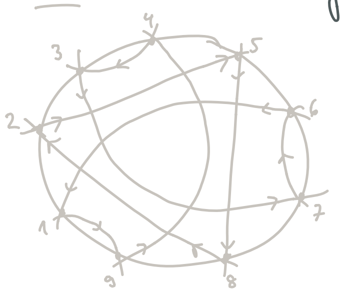
② * tilings of surfaces

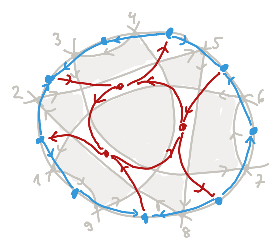
} vertices for diag's, boundary segments
arrows for "rotations" inside △s, tiles

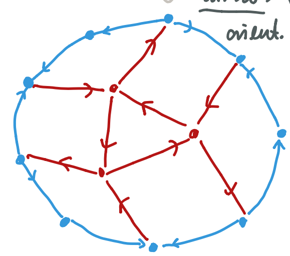
* alternating strand diagrams (Postnikov) on disk   or from plabic graphs

ex.: ↪ (k,n)-diagrams on disk (Scott) ; degenerations of them

} vertices for altern. regions
arrows from orient. of strands







③

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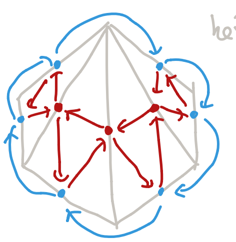
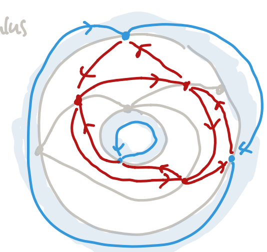
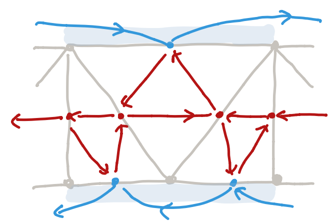
① triangulations of disks; of surfaces

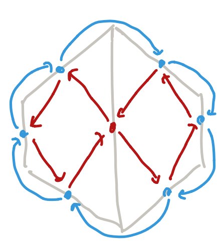
triangulated hexagon   annulus







② tilings of surfaces



④

Correspondences on P_n

triangul's/tilings —(b)→ alternating strand diagrams

Scott; BMartin

subset of

stellar replacement map (a)

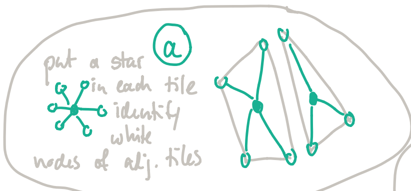
rhombic graphs

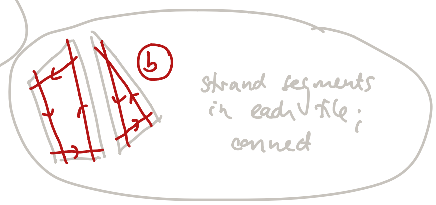
$\cap$

plabic graphs

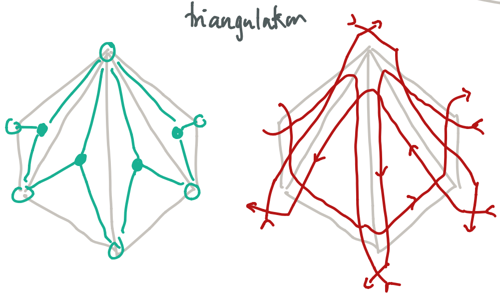
zig zag paths / rule of the road

reduced

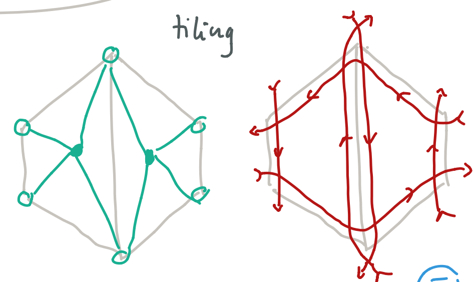




triangulation



tiling



5

Images under (a) and (b)

Rem.: The plabic graphs arising from tilings of P_n (are connected, fully reduced) have ≥ 1 black node

and * n white nodes on bdy, other nodes are black; * deg $(\bullet) \geq 3$ $\forall \bullet$;

* every closed face is a quadrilateral; * adj. edges at a white node are part of a quadrilateral.

Altern. strand diagram $\rightsquigarrow$ $\sigma \in S_n$ permutation.

n	3	4	5	6	7	6
# of permutations	1	2	7	26	100	404

$= d_{n-2}$

asympt. growth is 5,1918

appendix by M. Glick in BMartin '18

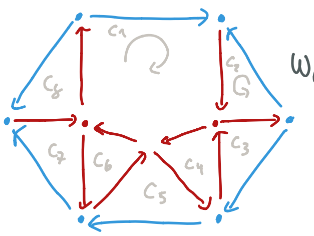
arXiv: 1601.05080

6

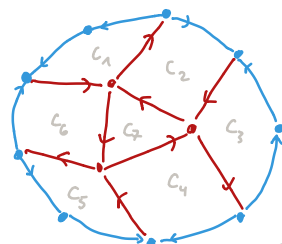
Algebras from dimer models

Q dimer model w. boundary; $S = S_+ \cup S_-$ the unit cycles of Q

$$W_Q := \sum_{\gamma \in S_+} \gamma \; - \sum_{\gamma \in S_-} \gamma$$



$W_Q = c_1 + c_3 + c_5 + c_7 - c_2 - c_4 - c_6 - c_8$



$W_Q = c_2 + c_4 + c_6 - c_1 - c_3 - c_5 - c_7$

$$A_Q := \widehat{\mathbb{C}Q} \Big/ \langle \{ \partial_\alpha W_Q : \alpha \text{ inner} \} \rangle$$

completed path alg. closure of rel's

For any inner arrow α

Relations: $c_i = \alpha p$, $c_j = \alpha q$

c_i, c_j the two cycles containing α

Then $p = q$

A_Q : dimer algebra of Q $\qquad e = e_1 + \dots + e_n$ from trivial paths at bdy vertices

$B_Q := e A_Q e$ the boundary algebra of Q.