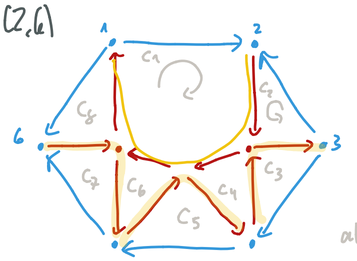
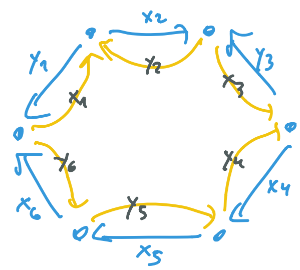
at end of talk I: Illustrate how to go from Q to the quiver of the algebra B_Q:

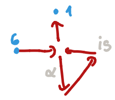
Q a (2,6) dimer



* keep blue arrows
* yellow arrows from paths.
* check that $xy = yx$ holds
* check $x^k = y^{n-k}$ hold
* check you can write all paths in terms of x, y



eg the path



is $y_6 x_6 x_1$ (composing left to right) (using relation for α)

Theorem (Jensen-King-Su) $\mathcal{F}_{k,n} := \{ B_{k,n}\text{-modules free over } Z(B_{k,n}) \}$ ← $\mathbb{C}[[t]]$ for $t := \sum x_i y_i$

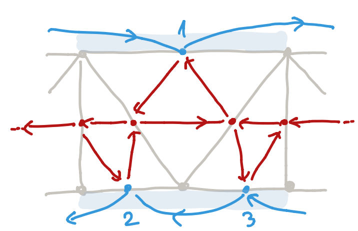
is a cluster category: an add. categorif. of Scott's cluster algebra structure of $\mathbb{C}[Gr(k,n)]$.

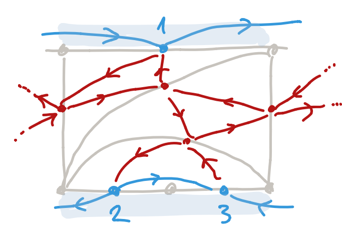
talk II

Thm (BKM) Q a (k,n)-dimer $\Rightarrow$ Q gives a cluster-tilting object T_Q for $\mathcal{F}_{k,n}$ and $\mathrm{End}(T_Q) \cong A_Q$.

Thm (BKM) $\mathcal{T}, \mathcal{T}'$ two triangul. of a marked surface $Q_{\mathcal{T}}, Q_{\mathcal{T}'}$ their quivers (dimer models w. bdy) $\Longrightarrow B_{Q_{\mathcal{T}}} \cong B_{Q_{\mathcal{T}'}}$

Ex. $\mathcal{T}, \mathcal{T}'$ triangul. of annulus $C_{2,1}$ with $Q_{\mathcal{T}}, Q_{\mathcal{T}'}$. Compute $B_{Q_{\mathcal{T}}}, B_{Q_{\mathcal{T}'}}$



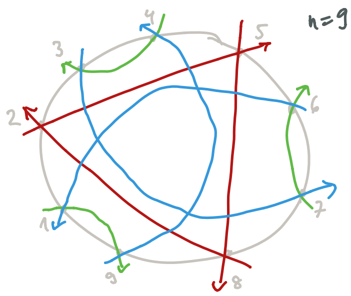


Orbifold diagrams (w. with Andrea Pasquali & Diego Velasco)

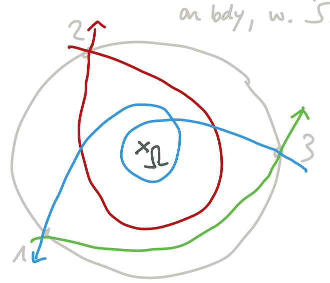
$\mathcal{G}$ alt. strand diagr. on P_n with d-fold rotational symmetry ⟶ $\mathcal{O} = \mathcal{G}/d$ orbifold diagram on O_{n_0} $(n = n_0 \cdot d)$

↖ disk, no pts on bdy, w. Ω orbifold of order d

$n=9,\ d=3$



⟶



permutation $\sigma = (258)(194376)$

permutation $\tau = (13)$

$O_{n_0}^{\Omega}$ disk w. no pts on bdy, Ω an orbifold pt of order $d \geq 2$

Def A weak orbifold diagram $\mathcal{O}$ on $O_{n_0}^{\Omega}$ of order d, of type $\tau \in S_{n_0}$

is a collection of n_0 curves c_i (strands) where $i \xrightarrow{c_i} \tau(i)$ the c_i don't go through Ω

* fin. many crossings, mult. 2, transversal
* alternating crossings
* "unoriented lenses" wind around Ω
* ∞ contain Ω or can be reduced

$\mathcal{O}$ is an orbifold diagram if the order d of Ω satisfies

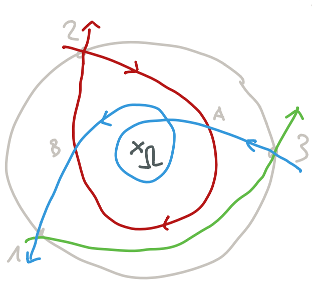
$$d > \max\{\underbrace{\max_c S(c)}_{m_S(\mathcal{O})}, \underbrace{\max L(c_1,c_2)}_{m_L(\mathcal{O})}\}$$

$S(c) := \max_{P \in c \cap c} |w(P)|$ winding # around Ω of the closed subcurve at P

$L(c_1,c_2) := \max |w(\text{loop})|$

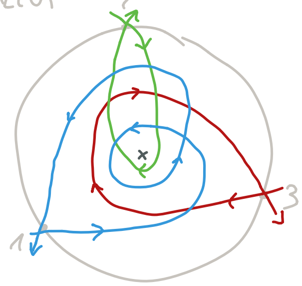
Ex.:

$\mathcal{O}_1$



$m_S(\mathcal{O}_1) = 1,\ m_L(\mathcal{O}_1) = 2$

$\mathcal{O}_2$



$m_S(\mathcal{O}_2) = 1,\ m_L(\mathcal{O}_2) = 2$

for $S(c)$: consider strand c w. self-inters. pt P; P determines closed subcurve of c

$L(c_1,c_2)$: closed loop formed by c_1, c_2^{-1} between A and B

$\mathcal{O}$ is called <u>Grassmannian</u> (of type (k,n)) if $\tau = \mathrm{id}$ and if $\exists\ w_+$, $0 < w_+ < d$, s.t. each strand has winding number w_+ or $w_+ - d$. (around Ω) $(n = n_0 \cdot d,\ k = w_+ \cdot d)$

<u>Ex.</u> $\mathcal{O}_2$ is of type $(3,9)$

<u>Remark</u>

$$\underset{\text{deg. } d}{\mathcal{O}^{\Omega}_{n_0}} \qquad \mathcal{O} \underset{g,\ /d}{\overset{\text{symd},\ f}{\rightleftarrows}} \mathcal{P} \quad \text{on } P_{n_0 d} \text{ with } d\text{-symmetry}$$

<u>Labels for orbifold diagrams</u>

assign labels from $\{1, \dots, n_0 d\}$ to $\mathcal{O}$

$*_1$ $\mathcal{I}$ the labels of symd$(\mathcal{O})$; $\mathcal{I}_{\mathcal{O}} := \mathcal{I}/n_0$ where $\{i_1, \dots, i_s\} \sim \{j_1, \dots, j_s\}$ if $\exists\ j$ s.t. $\{j_1, \dots, j_s\} = \{i_1 + jn_0, \dots, i_s + jn_0\}$

$*_2$ <u>algorithm</u>

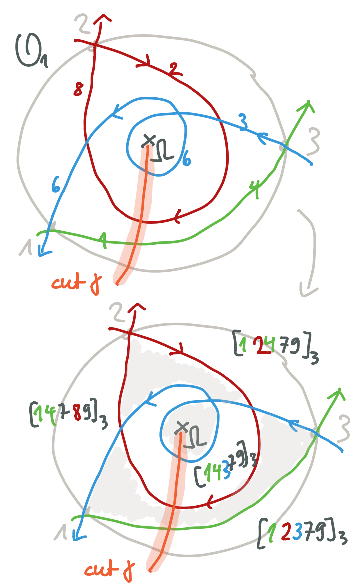
① Draw a cut γ from Ω to boundary segment $(n_0, 1)$

② If γ cuts c_i, write on segments
$\begin{cases} i,\ i+n_0,\ i+2n_0, \dots & \text{if } c_i \circlearrowleft \\ i,\ i-n_0,\ i-2n_0, \dots & \text{if } c_i \circlearrowright \end{cases}$ (reduce mod n)

③ altern. regions get label $i + mn_0$ if they are to left of segment $i + mn_0$ $(m \in \mathbb{Z})$

⚠ For alt. regions cut by γ: only label anti clockw. of γ

④ Add labels: if $j \in \{1, \dots, n = dn_0\}$ is <u>not</u> on any segment <u>and</u> if Ω is to the left of c_{j_0} (j_0 red. of j mod n), add j to <u>all</u> alt. regions.



Quiver with potential for orbifold diagrams : $\mathcal{O}$ an orbifold diagram, $\mathrm{ord}(\Omega)=d$

quiver

$Q_{\mathcal{O}}$: $\begin{cases} \text{vertices for altern. regions} & \cup\{v_1,\dots,v_d\} \\ \text{arrows as for altern. str. diagrams} & \cup \left\{ \text{arrows for } v_i \text{ w. every neighb. alt. region}, \; i=1,\dots,d \right\} \end{cases}$ if Ω is in an altern. region

← these form r unit cycles $c_i^{(1)},\dots,c_i^{(r)}$ for some r, $\forall i$

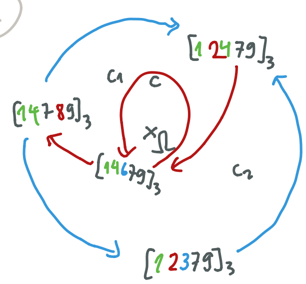
potential

$$W_{\mathcal{O}} := \begin{cases} \frac{1}{d}\,\mathrm{sgn}(c)c^d + \sum_{c'\neq c} \mathrm{sgn}(c')c' & \text{if } \Omega \text{ is in an oriented face, bdd by unit cycle } c \\ \sum_{i=1}^{d} \left(\xi^i\,\mathrm{sgn}(c_i^{(r)})c_i^{(r)} + \sum_{j\neq r} \mathrm{sgn}(c_i^{(j)})c_i^{(j)} \right) + \sum_{c'\neq c_i^{(j)}\,\forall i,j} \mathrm{sgn}(c')c' & \text{if } \Omega \text{ in altern. face} \end{cases}$$

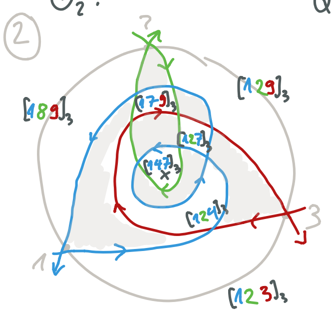
ξ a primitive d^{th} root of 1

Giovannini Pasquali '19   Giovannini-Pasquali-Plamondon '19

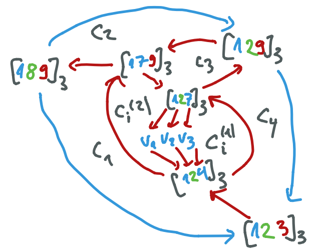
Ex ① $Q_{\mathcal{O}_1}$; $W_{\mathcal{O}_1} = c_1 - c_2 - \frac{1}{3}c^3$



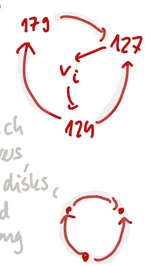
② $\mathcal{O}_2$:



$Q_{\mathcal{O}_2}$; $W_{\mathcal{O}_2} = \begin{array}{lll} -c_1 - c_3 & -c_1^{(1)} & +\xi c_1^{(2)} \\ +c_2 + c_4 & -c_2^{(1)} & +\xi c_2^{(2)} \\ & -c_3^{(1)} & +\xi c_3^{(2)} \end{array}$



3 such quivers on disks, glued along



skew-group Algebras & categories from orbifold diagrams

$\mathcal{O}$ orbifold diagram, order d. G = cyclic gp gen. by rotations by $2\pi/d$ G acts

$\mathcal{P} := \mathrm{symd}(\mathcal{O})$; $Q_{\mathcal{P}}$ its dimer model

$A_{\mathcal{O}} := A(Q_{\mathcal{O}}, W_{\mathcal{O}})$, $B_{\mathcal{O}} = eA_{\mathcal{O}}e$

Then $A_{\mathcal{O}} \sim A_{Q_{\mathcal{P}}} * G$

$B_{\mathcal{O}} \sim B_{Q_{\mathcal{P}}} * G$

Recall : $Q_{\mathcal{P}}$ a (k,n)-dimer, $B_{Q_{\mathcal{P}}} \cong B_{k,n}^{op}$ $(n = n_0 d)$

$$B_G := B_G(n_0,k,n) = \mathbb{C}\Pi_{n_0} \Big/ \left\{ \begin{matrix} xy - yx \\ x^k - y^{n-k} \end{matrix} \right\}$$

$\rightsquigarrow$ $B_{\mathcal{O}} \cong B_G^{op}$

can define modules for the algebra B_G !

Rank 1 modules for B_G

I a k-subset of $\{1,\dots,n\}$
$[I]_{n_0}$ it's equiv. class

$$L_{[I]_{n_0}} = \bigoplus_{i=0}^{d-1} \bigoplus_{j=1}^{n_0} \mathbb{C}[[t]] \quad \text{as v.space}$$

For $[I]_{n_0} \neq [J]_{n_0}$, $L_{[I]_{n_0}} \not\cong L_{[J]_{n_0}}$

$$T_\Theta := \bigoplus_{[I]_{n_0} \in \mathcal{I}_\Theta} L_{[I]_{n_0}}$$

Theorem (B-Pasquali-Velasco '20)

$$A_\Theta \cong \mathrm{End}_{B_G}(T_\Theta)$$

Rem.: ① In BKM, $Q_\mathfrak{g}$ gives a cluster tilting obj. of $\underline{\mathcal{F}_{k,n}}$. Demonet (PhD, §2.2.4): the skew group category $\underline{\mathcal{F}_{k,n}} * G$ is Frobenius, stably 2-CY and T_Θ is in it; it is cluster-tilting. ② We do not have a concrete description of $\mathcal{F}_{k,n} * G$ as (equiv. to) a subcat. of $\mathrm{mod}(B_G)$.

Quivers, orbifolds & their plabic graphs (for running examples)