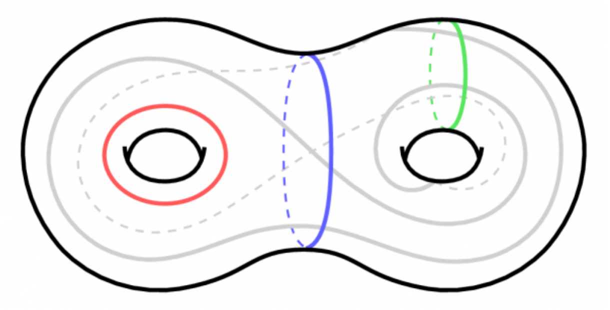
Curved Dimers

Raf Bocklandt

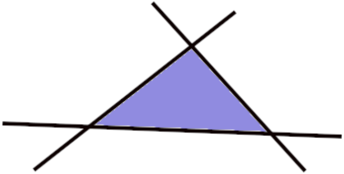
August 13, 2020

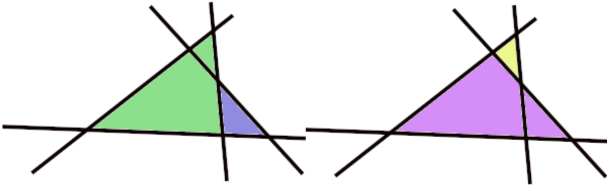
The Fukaya category of a closed surface



- Objects: closed curves on the surface,
- Morphisms: Linear combinations of intersection points,

Products



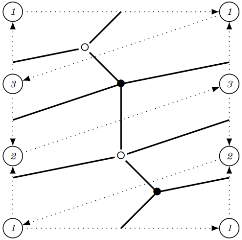


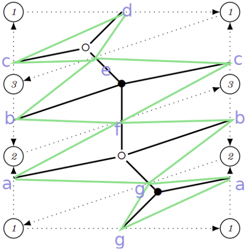
Problems

- Monogons, digons, polygons
- Signs
- Self-intersections
- Convergence
- Huge

Different approaches to solve all these problems (Seidel, Abouzaid, Fukaya, etc).

An approach using dimers





The *gentle A_∞-algebra* of an arc collection is the path algebra of $Q_{\mathcal{A}}$ divided by the ideal generated by the face like paths.

$$\mathtt{Gtl}^{\pm}\mathcal{A} = \frac{\mathbb{C}Q_{\mathcal{A}}}{\langle \alpha\beta | \alpha\beta \in P_{-} \rangle}$$

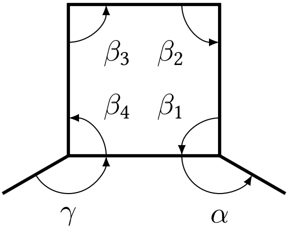
We give the algebra a $\mathbb{Z}_2$-grading using the rule below.

An approach using dimers: The higher products

On $\mathtt{Gtl}^{\pm}\mathcal{A}$ we put an A_∞-structure defined by the rule that

$$\mu(\alpha\beta_1, \ldots, \beta_k) = \alpha \text{ and } \mu(\beta_1, \ldots, \beta_k\gamma) = (-1)^{\deg\gamma}\gamma$$

if $\beta_1, \ldots, \beta_l$ are the consecutive angles of an immersed polygon without internal marked pounts bounded by arcs.



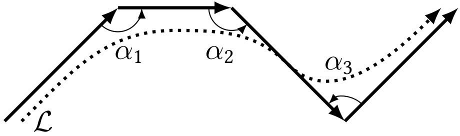
A *twisted complex* A over $\mathrm{C}^\bullet$ consists of a pair (M, δ) where $M = \oplus_i A_i[k_i]$ is a direct sum of shifted objects and δ is a degree 1 element in $\mathrm{Hom}(M, M)$. Additionally we assume that δ is *strictly lower triangular* satisfies the *Maurer-Cartan equation*:

$$\mu_1(\delta) + \mu_2(\delta, \delta) + \mu_3(\delta, \delta, \delta) + \cdots = 0.$$

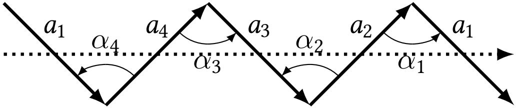
Given a sequence of twisted complexes $(M_0, \delta_0), \cdots, (M_k, \delta_k)$ we can introduce a twisted k-ary product by taking a sum over all possible ways to insert $\delta' s$ between the entries

$$\tilde{\mu}_k(a_1, \cdots, a_k) = \sum_{m_0, \cdots, m_k \geq 0} \mu_\bullet(\underbrace{\delta_0, \cdots, \delta_0}_{m_0}, a_1, \cdots, a_k, \underbrace{\delta_k, \cdots, \delta_k}_{m_k}).$$

$$\mathrm{C}^\bullet \subset \mathrm{Tw}\, \mathrm{C}^\bullet$$



$$(L, \delta) := \left(\bigoplus a_j, \sum_{j=1}^{k-1} \alpha_u \right)$$



$$(B, \delta) = \left(\bigoplus_{j=1}^{k} a_j, \delta = \sum_{j=1}^{k} \lambda_j \alpha_j \right)$$

Deformation theory

- A *deformation of an A_∞-algebra A, μ* is a $k[[\hbar]]$-linear $\mathbb{Z}_2$-graded curved A_∞-structure $\mu_\hbar$ on $A[[\hbar]]$ that reduces to A, μ if we quotient out $\hbar$.
- Deformation theory of (A, μ) is described by its Hochschild cohomology.
- A *deformed twisted complex* is a pair

$$M = (\oplus_i A_i[j_i], \delta + \epsilon)$$

where $(\oplus_i A_i[j_i], \delta)$ is an ordinary twisted complex and $\epsilon \in \hbar \mathrm{End}(\oplus_i A_i[j_i])[[\hbar]]$. The curvature of M is $\mu(\delta + \epsilon) + \mu(\delta + \epsilon, \delta + \epsilon) + \dots$

- The deformed twisted complexes form a defomation of (a category equivalent to) the twisted complexes of A.

See also Lowen-Van den Berg (https://arxiv.org/abs/1505.03698), FOOO.

Deformation of gentle A_∞-algebras

Theorem

The Hochschild cohomology of the gentle A_∞-algebra $\mathtt{Gtl}^{\pm}\mathcal{A}$ is equal to

- $\mathsf{HH}^0(\mathtt{Gtl}^{\pm}\mathcal{A}) = \bigoplus_{m\in M} \mathbb{C}[\ell_m]\ell_m\partial_m \oplus \mathbb{C}^{n+2g-1}$,
- $\mathsf{HH}^1(\mathtt{Gtl}^{\pm}\mathcal{A}) = \frac{\mathbb{C}[\ell_m|m\in M]}{(\ell_i\ell_j|i\neq j,i,j\in M)}$.

Proof.

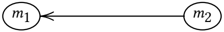
Either via direct computation using Bardzel's bimodule resolution of $\mathtt{Gtl}^{\pm}\mathcal{A}$ or using mirror symmetry and matrix factorizations (Lin-Pomerleano/Wong). □

Question

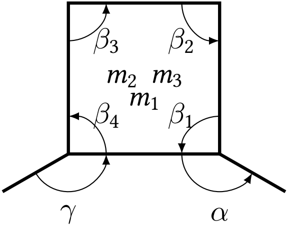
Can we describe the deformations explicitely?
(Joint with Van de Kreeke)

For each deformation class of $\mathtt{Gtl}^{\pm}\mathcal{A}$ we can find a nice curved A_∞-algebra deformation of $\mathtt{Gtl}^{\pm}\mathcal{A}$ (over $k[\hbar]$ instead of $k[[\hbar]]$). E.g. if $f = \sum_i \lambda_i \ell_i$ with $\lambda_i \in (\hbar)$ then we set

$$\mu_0(a) = \lambda_{m_1}\ell_{m_1} + \lambda_{m_2}\ell_{m_2} \text{ if}$$



and if $\beta_1, \ldots, \beta_l$ are the consecutive angles of an immersed polygon with internal marked points bounded by arcs.



we set

$$\mu(\alpha\beta_1, \ldots, \beta_k) = \lambda_1\lambda_2\lambda_3\alpha \text{ and } \mu(\beta_1, \ldots, \beta_k\gamma) = (-1)^{\deg\gamma}\lambda_1\lambda_2\lambda_3\gamma.$$

Deforming strings and bands

- After deforming the gentle algebra, every string and band becomes curved.
- Band objects that do not enclose a disk are isomorphic to an uncurved object.
- In general we cannot keep $\hbar$ inside $k[\hbar]$ if we go to the twisted completion.
- Solution: look at a fixed number of band objects depending on the dimer.
- Behaviour depends on the genus of the surface and dimer.