THE EFFECT OF MICROSCOPIC GAP DISPLACEMENT ON THE CORRELATION OF GAPS IN DIMER SYSTEMS

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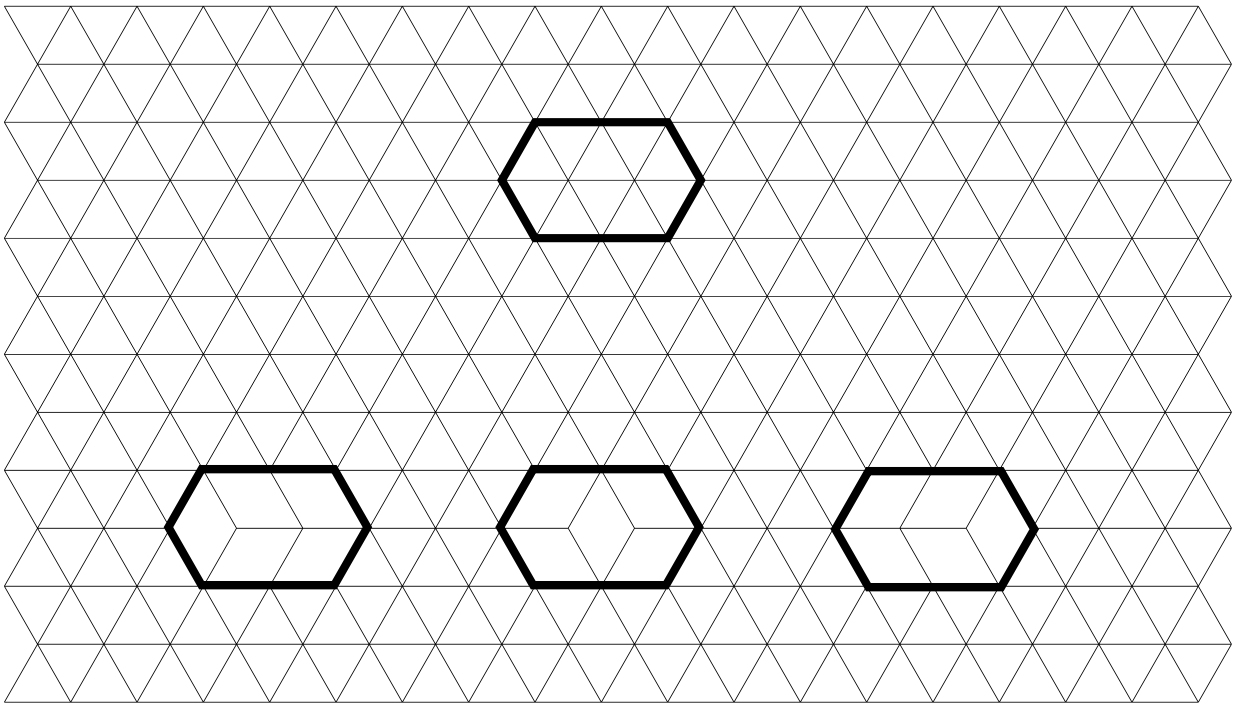
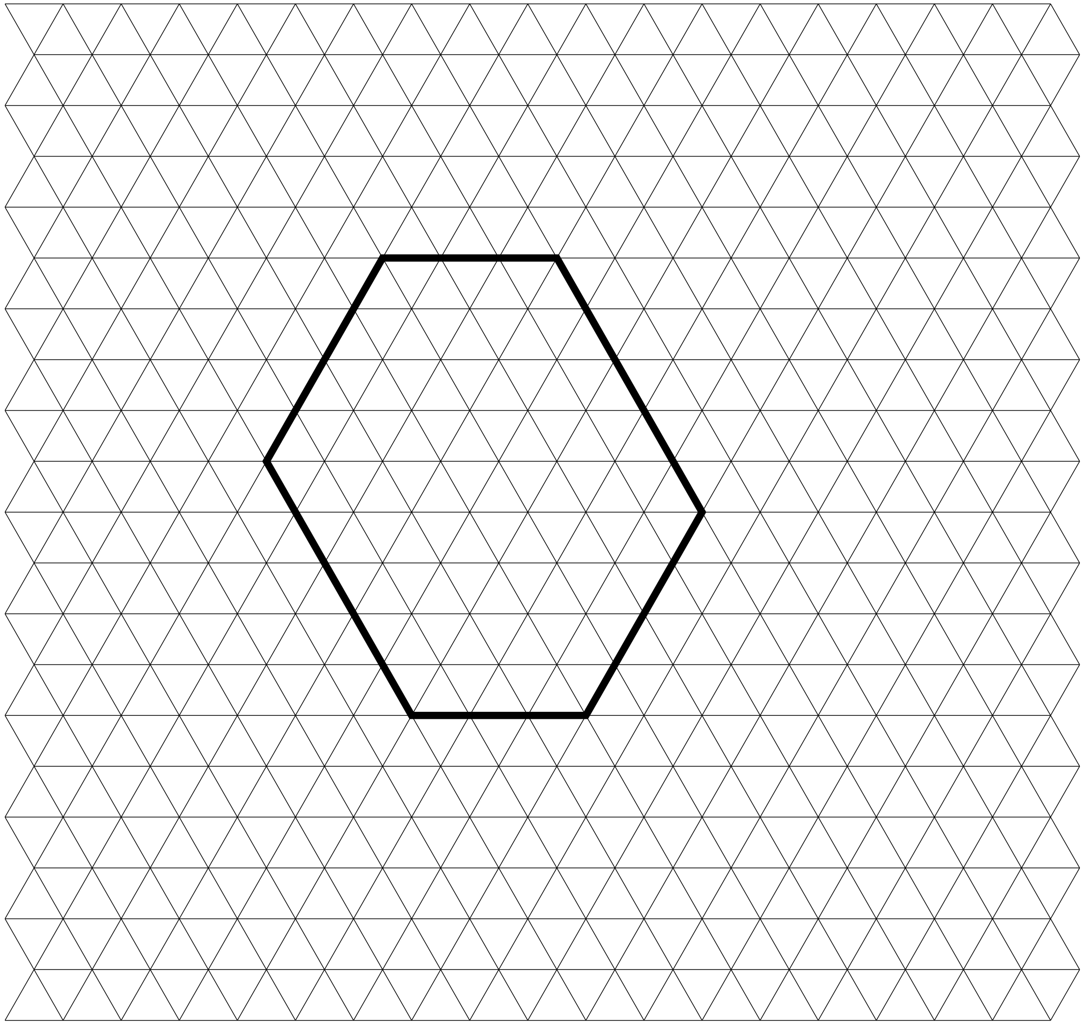
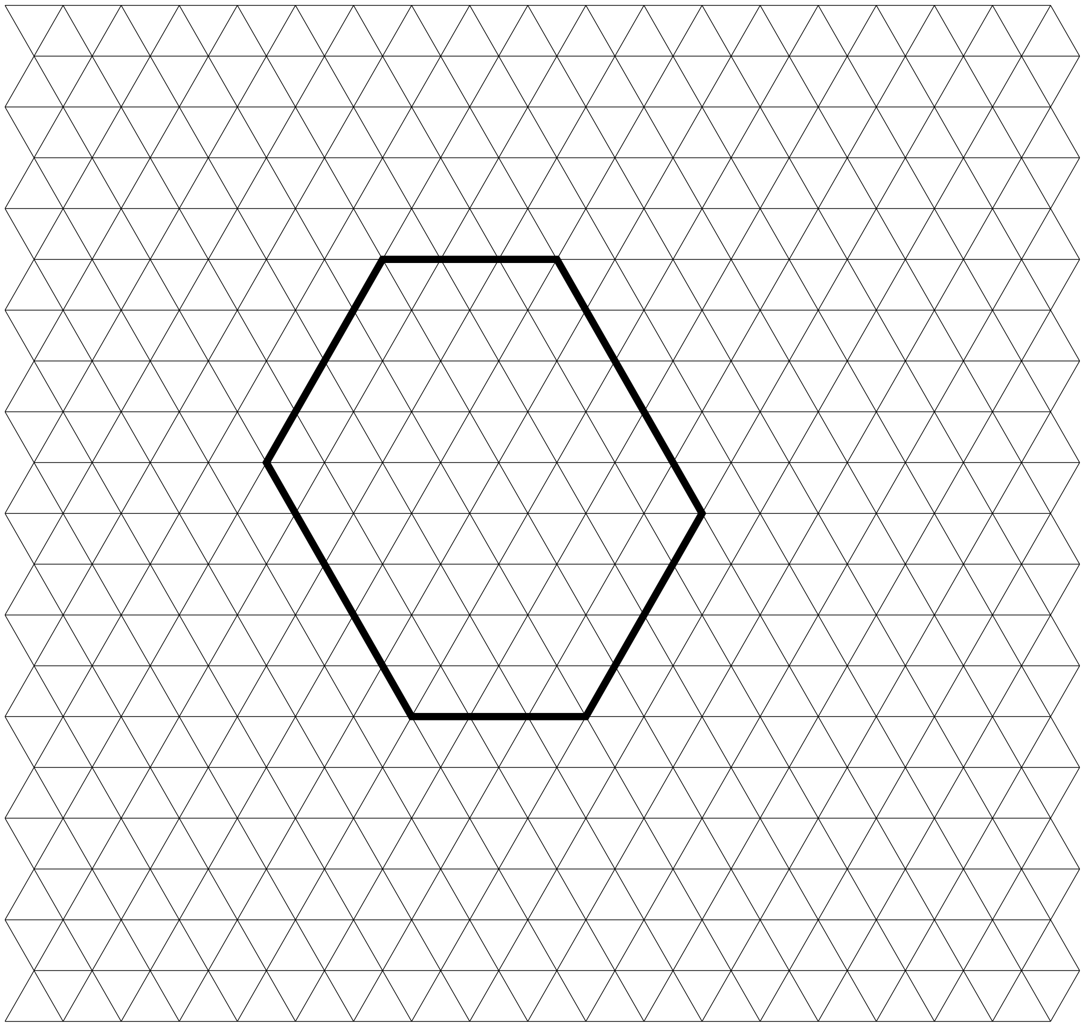


$H_{1,2,1}$ and its three lozenge tilings

$$\mathrm{M}(H_{1,2,1}) = 3$$



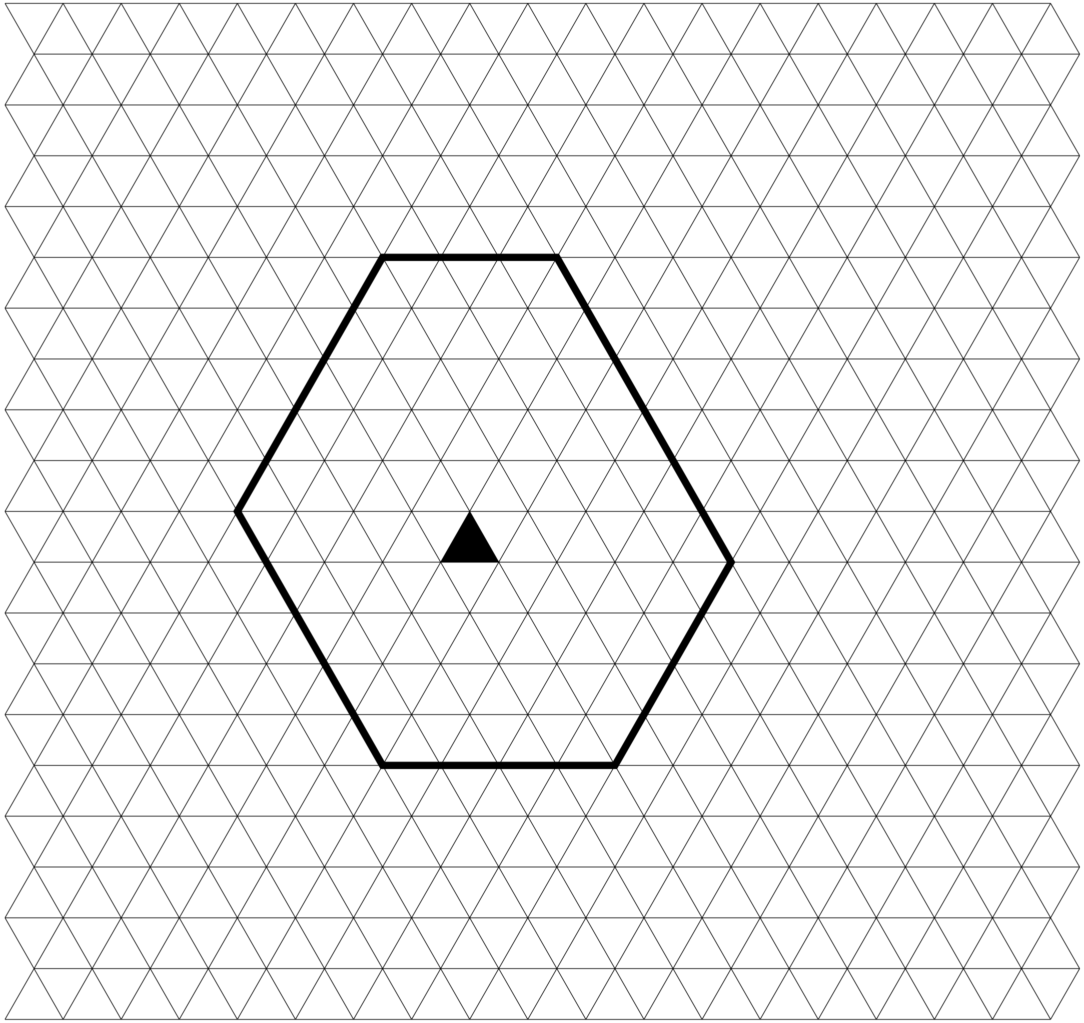
The hexagon $H_{3,5,4}$



The hexagon $H_{3,5,4}$

$$\frac{\mathrm{H}(3)\ \mathrm{H}(4)\ \mathrm{H}(5)\ \mathrm{H}(3+4+5)}{\mathrm{H}(3+4)\ \mathrm{H}(3+5)\ \mathrm{H}(4+5)} = 116424 \ \ \text{tilings}$$

$(\mathrm{H}(n) := 0!\,1!\cdots(n-1)!)$

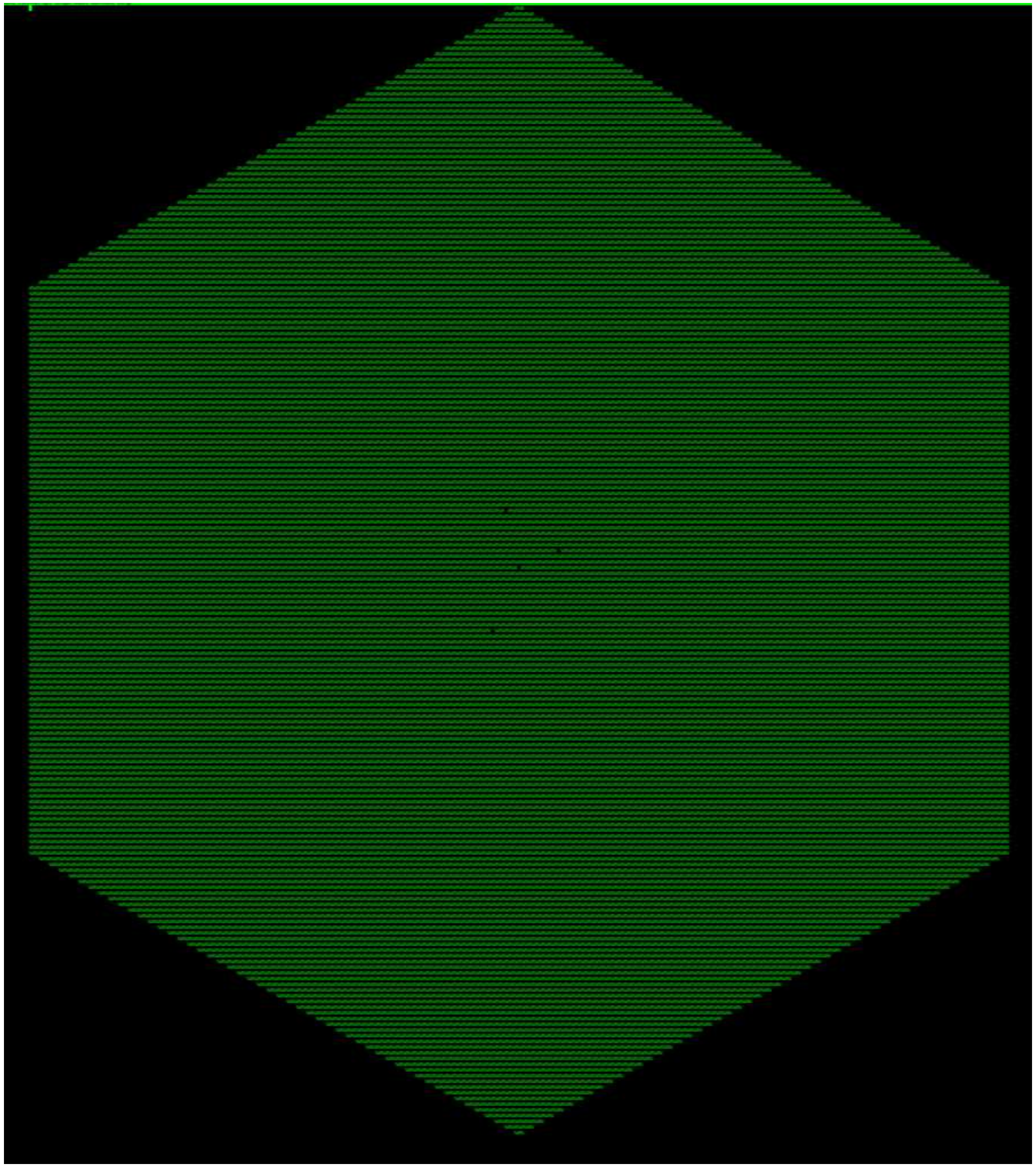


How many tilings?

1000000

Think of very large regions with a few holes near the “center”

We would like to understand how the number of tilings changes as the holes move around the center



3 numbers: This, center shifted SE, center shifted NW — relative changes?

If the numbers are a_0, a_1 and a_2, we have

$$\frac{a_1 - a_0}{a_0} = 0.1174023961...$$

$$\frac{a_2 - a_0}{a_0} = -0.1144518996...$$

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Why nearly opposite?

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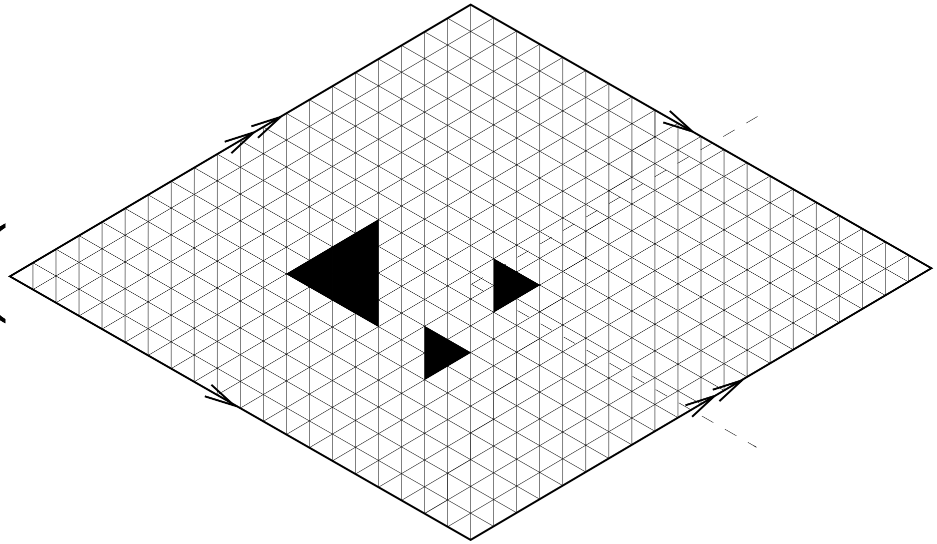
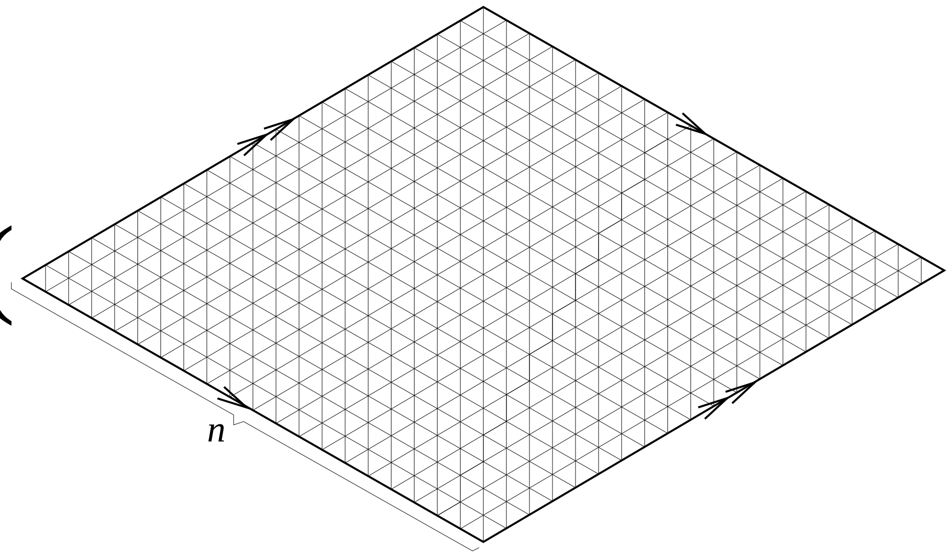
Why nearly opposite?

(These numbers have more than 3000 digits!)

The correlation ω

M ()

$\omega_0 := \lim_{n\to\infty}$ ――――――――――――――――

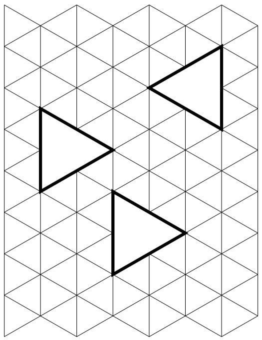
M ()

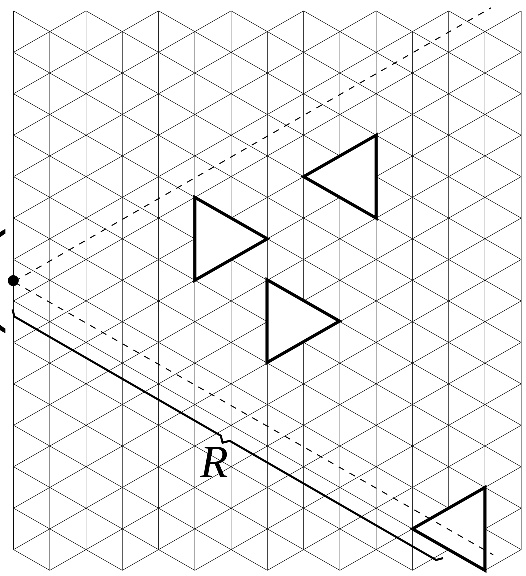
Note: This only works when $\# \triangleright - \# \triangleleft = 0$.

Extend to arbitrary unions of triangles of side 2

If $\# \triangleright - \# \triangleleft = 0$, define $\omega := \omega_0$.

If $\# \triangleright - \# \triangleleft = 2$:

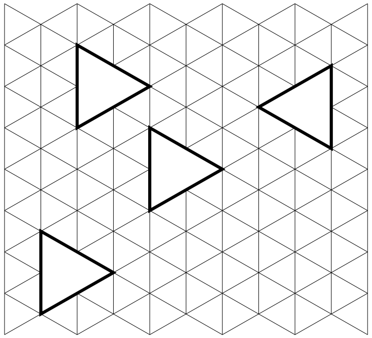
$\omega\Big($  $\Big) :=$

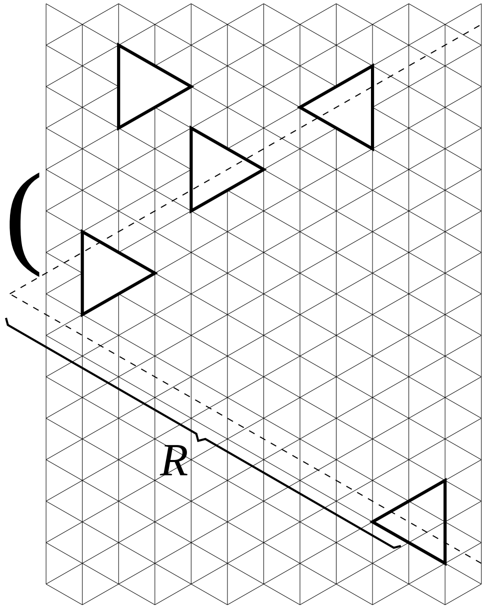
$\frac{1}{c_0} \lim_{R\to\infty} R^2 \;\; \omega\Big($ 

$\Big)$

where $c_0 = \frac{\sqrt{3}}{2\pi}$

If $\# \triangleright - \# \triangleleft = 4$:

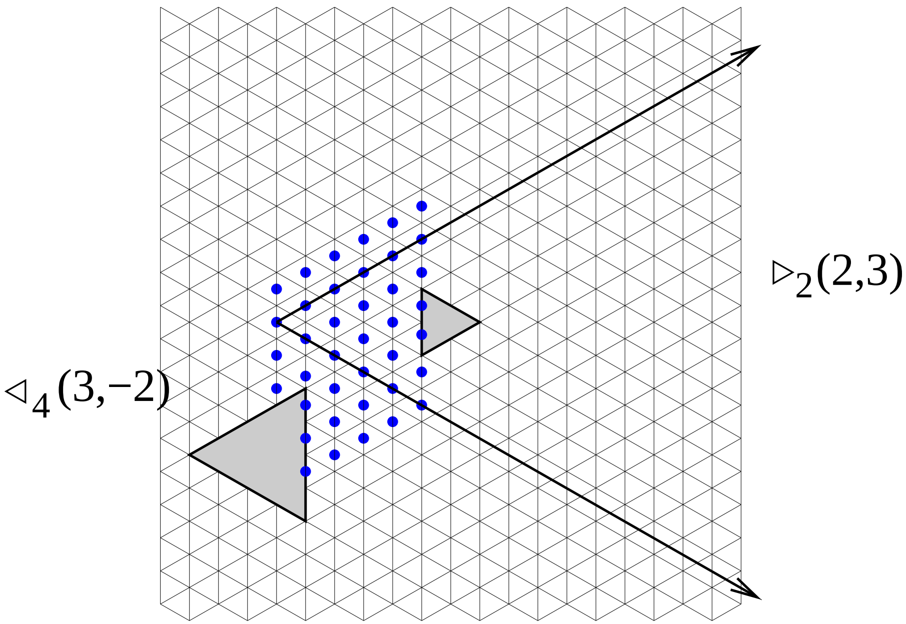
$\omega\Big($  $\Big) :=$

$\dfrac{1}{c_0} \lim\limits_{R \to \infty} R^4 \ \omega\Big($ 

$\Big)$

Charge of a hole:

$\mathrm{q}(O) := \#(\triangleright\text{'s in } O) - \#(\triangleleft\text{'s in } O)$



Translation of a hole:

- Base point: topmost and leftmost marked point in O
- $O(x, y)$: the translation of O that brings its base point to the point (x, y)

- Consider $x_1^{(R)}, \ldots, x_n^{(R)}, y_1^{(R)}, \ldots, y_n^{(R)} \in \mathbb{Z}$ with

$$\lim_{R\to\infty} \frac{x_i^{(R)}}{R} = x_i, \lim_{R\to\infty} \frac{y_i^{(R)}}{R} = y_i$$

- Assume the (x_i, y_i)'s are distinct.

Theorem 1 (C., 2009) Suppose O_i is either of type $\triangleright_k$ or of type $\triangleleft_k$, with k even, for $i = 1, \ldots, n$. Then if $O_i^{(R)} = O_i(x_i^{(R)}, y_i^{(R)})$,

$$\omega(O_1^{(R)}, \ldots, O_n^{(R)}) \sim \prod_{i=1}^{n} \omega(O_i) \prod_{1\leq i<j\leq n} \mathrm{d}(O_i^{(R)}, O_j^{(R)})^{\frac{1}{2}\,\mathrm{q}(O_i)\,\mathrm{q}(O_j)}, \quad R \to \infty,$$

and $\omega(\triangleright_{2s}) = \dfrac{3^{s^2/2}}{(2\pi)^s}\,[0!\,1!\,\cdots\,(s-1)!]^2$.

Conjecture 1 (C., 2008) Statement of theorem holds for *any* holes $O_1, \dots, O_n$.

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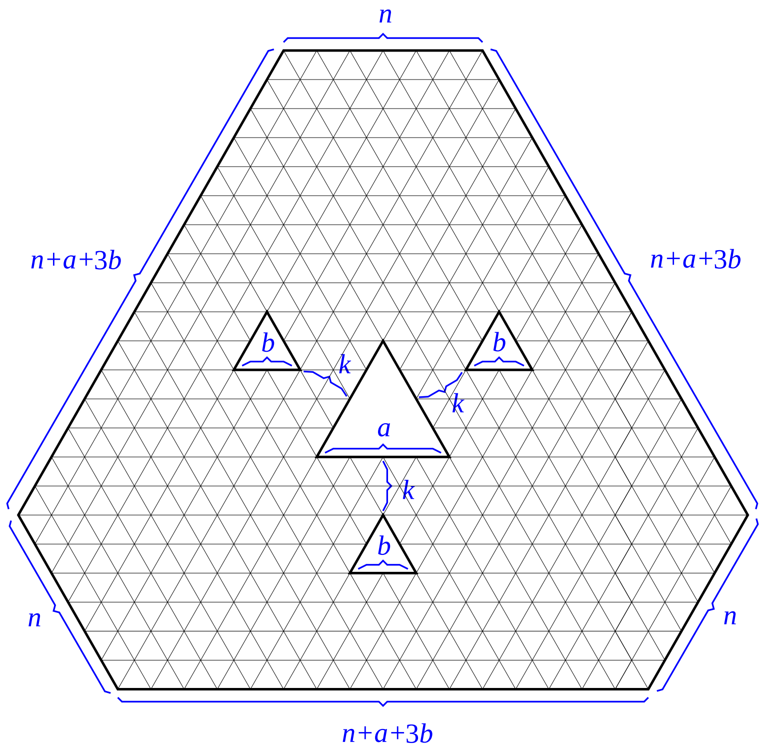
Note: What about $\omega(\triangleright_{2s+1})$?



$$\overline{\mathrm{M}}(S_{n,a,b,k}) := \frac{\mathrm{M}(S_{n,a,b,k})}{\mathrm{M}(S_{n,a,b,0})}, \quad \overline{\mathrm{M}}_r(S_{n,a,b,k}) := \frac{\mathrm{M}_r(S_{n,a,b,k})}{\mathrm{M}_r(S_{n,a,b,0})}$$

Conjecture 2 (C. and Fischer, 2019) For non-negative integers n, a, b and k with a even we have

$$\frac{\overline{\mathrm{M}}(S_{n,a,b,k})}{\overline{\mathrm{M}}_r(S_{n,a,b,k})^3} = \left[\prod_{i=1}^{k} \frac{(a+6i-4)(a+3b+6i-2)}{(a+6i-2)(a+3b+6i-4)}\right]^2.$$

This implies that, if the electrostatic hypothesis conjecture holds, we must have

$$\omega(\triangleright_k) = \frac{3^{k^2/8}}{(2\pi)^{k/2}} \left[G\left(\frac{k}{2}+1\right)\right]^2, \quad \text{all } k \geq 0.$$

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Amusing note: The assumption that $\omega(\triangleright_k)$ is given by the above formula for odd k, together with the two conjectures above, imply that if A is the Glaisher-Kinkelin constant, which is given by

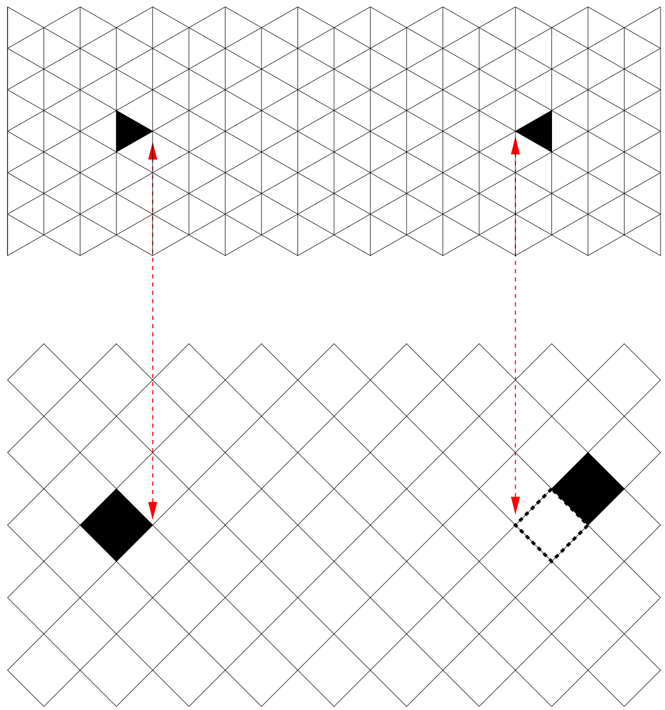
$$\lim_{n\to\infty} \frac{0!\,1!\,\cdots\,(n-1)!}{n^{\frac{n^2}{2}-\frac{1}{12}}\,(2\pi)^{\frac{n}{2}}\,e^{-\frac{3n^2}{4}}} = \frac{e^{\frac{1}{12}}}{A},$$

then

$$G\left(\frac{1}{2}\right) = \frac{e^{1/8}2^{1/24}}{A^{3/2}\pi^{1/4}},$$

— which is in fact the correct value!

Another consequence: Squaring the hexagonal lattice



- Line up removed unit triangles and the removed unit squares as shown
- Both correlations decay to zero like $c/\sqrt{\text{distance}}$, with $c = e^{1/2} 2^{-5/6} A^{-6}$

Recall:

- $x_1^{(R)}, \ldots, x_n^{(R)}, y_1^{(R)}, \ldots, y_n^{(R)} \in \mathbb{Z}$, $\lim_{R\to\infty} \dfrac{x_i^{(R)}}{R} = x_i$, $\lim_{R\to\infty} \dfrac{y_i^{(R)}}{R} = y_i$

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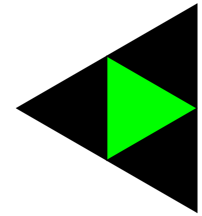
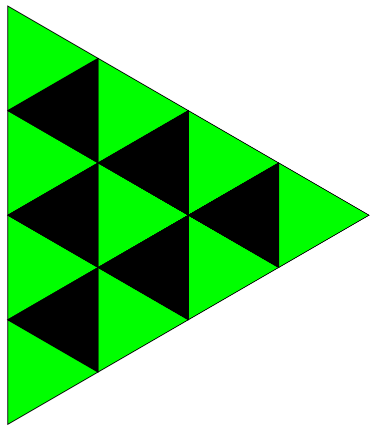
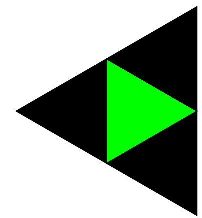
and $\omega(\triangleright_{2s}) = \dfrac{3^{s^2/2}}{(2\pi)^s}\,[0!\,1!\,\cdots\,(s-1)!]^2$.

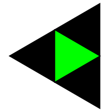
View this as a fine mesh limit

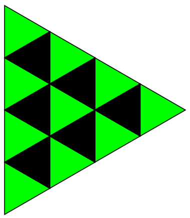
- Lattice spacing approaches zero
- Holes shrink to the points $(x_1, y_1), \ldots, (x_n, y_n)$

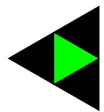


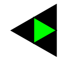


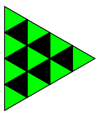


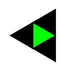












Then Theorem 1 gives the effect of **macroscopic gap displacements** on the correlation:

If arrange for O_1 to shrink to point (x'_1, y'_1) instead, then

$$\frac{\omega(Q_1^{(R)}, \ldots, O_n^{(R)})}{\omega(O_1^{(R)}, \ldots, O_n^{(R)})} \to \frac{\prod_{j>1}((x_j - x'_1)^2 + (x_j - x'_1)(y_j - y'_1) + (y_j - y'_1)^2)^{\frac{1}{4}\,\mathrm{q}(O_1)\,\mathrm{q}(O_j)}}{\prod_{j>1}((x_j - x_1)^2 + (x_j - x_1)(y_j - y_1) + (y_j - y_1)^2)^{\frac{1}{4}\,\mathrm{q}(O_1)\,\mathrm{q}(O_j)}},$$

where $Q_1^{(R)}$ is the new R-th translate of O_1.

(In our coordinate system, $\mathrm{d}((a,b),(c,d)) = \sqrt{(a-b)^2 + (a-b)(c-d) + (c-d)^2}$.)

But what if we displace a gap just a fixed amount, say 1 unit?

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If we knew *exactly* the value of ω, we would know this.

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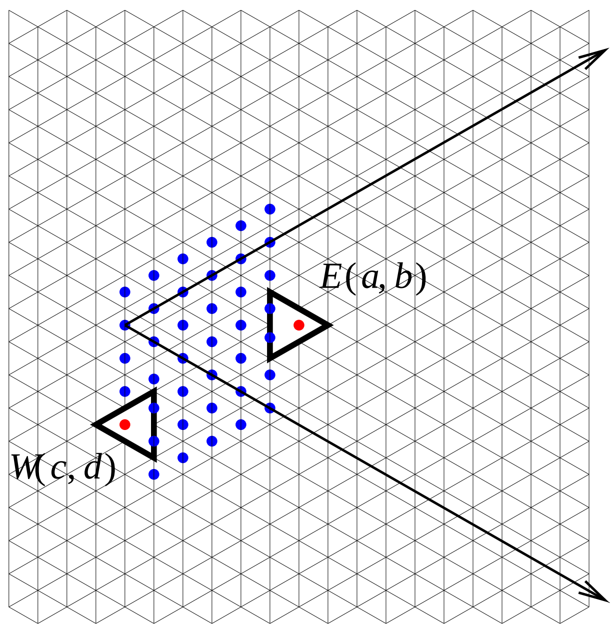

A simple case: Two E's

$$\omega(E(a,b),E(c,d)) = \frac{3}{4\pi^2}\begin{cases} (a-c)^2+(a-c)(b-d)+(b-d)^2, & 3|(a-b)-(c-d) \\ (a-c)^2+(a-c)(b-d)+(b-d)^2-1, & \text{otherwise} \end{cases}$$

Three E's

If $3|a-b, c-d, e-f$,

$$\omega(E(a,b),E(c,d),E(e,f)) = \left(\frac{\sqrt{3}}{2\pi}\right)^3 (t_1+t_2+t_3),$$

$$t_1 = \prod \left\{(a-b)^2 + (a-c)(b-d) + (b-d)^2\right\}$$

$$t_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & e & c \\ b & d & f \end{vmatrix} \sum \{ad(a+d) - bc(b+c) + 2(ac-bd)(a-c+b-d)\}$$

$$t_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & e & c \\ b & d & f \end{vmatrix}^2$$

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So if the E's are collinear, $t_2 = t_3 = 0$ — "exactness"!

- Exactness holds in fact for any number of collinear $E(a_i, b_i)$'s with $3|a_i - b_i$.

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- Even for 3 E's, outside the case $3|a_i - b_i$ exact value is not nice — don't even have exactness for collinear holes!

Question. What is relative change in ω if we displace one hole microscopically?

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Example: 2 E's, 1 W

What is the $R \to \infty$ asymptotics of

$$T_{\alpha,\beta}^{E(Ra_0,Rb_0)} := \frac{\omega(E(Ra_0+\alpha, Rb_0+\alpha), E(Ra, Rb), W(Rc, Rd))}{\omega(E(Ra_0, Rb_0), E(Ra, Rb), W(Rc, Rd))} - 1?$$

Determinant formula for the correlation ω

- $P(x,y) := \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{e^{ix\theta}e^{iy\phi}d\theta\, d\phi}{1+e^{-i\theta}+e^{-i\phi}}$

- $P(-3r-1+a,-1+b) \sim \sum_{s=0}^{\infty}(3r)^{-s-1}U_s(a,b)$

- $A(x,y) := \begin{bmatrix} P(x-1,y-1) & P(x-2,y) \\ P(x,y-2) & P(x-1,y-1) \end{bmatrix}$

- $B_s(x,y) := \begin{bmatrix} U_s(x,y) & U_s(x-1,y+1) \\ U_s(x+1,y-1) & U_s(x,y) \end{bmatrix}$

Proposition. For $m \geq n$,

$$\omega(E(a_1,b_1),\ldots,E(a_m,b_m),W(c_1,d_1),\ldots,W(c_n,d_n))$$

is $\left(\frac{\sqrt{3}}{2\pi}\right)^{n-m}$ times the absolute value of the determinant of the matrix

$$\begin{bmatrix} A(a_1-c_1,b_1-d_1) & A(a_1-c_2,b_1-d_2) & \cdots & A(a_1-c_n,b_1-d_n) & B_0(a_1,b_1) & B_1(a_1,b_1) & \cdots & B_{m-n-1}(a_1,b_1) \\ A(a_2-c_1,b_2-d_1) & A(a_2-c_2,b_2-d_2) & \cdots & A(a_2-c_n,b_2-d_n) & B_0(a_2,b_2) & B_1(a_2,b_2) & \cdots & B_{m-n-1}(a_2,b_2) \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ A(a_m-c_1,b_m-d_1) & A(a_m-c_2,b_m-d_2) & \cdots & A(a_m-c_n,b_m-d_n) & B_0(a_m,b_m) & B_1(a_m,b_m) & \cdots & B_{m-n-1}(a_m,b_m) \end{bmatrix}$$

Lemma. Let $\zeta = e^{2\pi i/3}$, $Df(x) = f(x+1) - f(x)$. Then

$$U_s(a,b) = -\frac{i}{2\pi} \left[\zeta^{a-b-1}(1 - D\zeta^{-1})^{-b} - \zeta^{-a+b+1}(1 - D\zeta)^{-b}\right] (x^s)\Big|_{x=a+b-1} .$$

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So for instance

$$U_0(a,b) = -\frac{i}{2\pi}\left(\zeta^{a-b-1} - (1/\zeta)^{a-b-1}\right) = -\frac{\sqrt{3}}{2\pi}\begin{cases} 1, & a-b = 0 \pmod 3 \\ 0, & a-b = 1 \pmod 3 \\ -1, & a-b = 2 \pmod 3 \end{cases}$$

$$U_1(a,b) = -\frac{i}{2\pi}\left(\zeta^{a-b-1}(a+b-1+b/\zeta) - (1/\zeta)^{a-b-1}(a+b-1+b\zeta)\right)$$

$$= -\frac{\sqrt{3}}{2\pi}\begin{cases} a-1, & a-b = 0 \pmod 3 \\ b, & a-b = 1 \pmod 3 \\ -a-b+1, & a-b = 2 \pmod 3 \end{cases}$$

$$U_2(a,b) = -\frac{\sqrt{3}}{2\pi}\begin{cases} a^2-b^2-2a-b+1, & a-b = 0 \pmod 3 \\ 2ab+b^2-2b, & a-b = 1 \pmod 3 \\ -a^2-2ab+2a+3b-1, & a-b = 2 \pmod 3 \end{cases}$$

Then, assuming for simplicity $3|\alpha-\beta, a_0-b_0, a-b, c-d$, we have:

$$\omega(E(Ra_0+\alpha, Rb_0+\alpha), E(Ra, Rb), W(Rc, Rd)) = \left(\frac{\sqrt{3}}{2\pi}\right)^{-1}$$

$$\times \left|\det \begin{bmatrix} P(-R(c-a_0)-1+\alpha,-R(d-b_0)-1+\beta) & P(-R(c-a_0)-2+\alpha,-R(d-b_0)+\beta) & 1 & 0 \\ P(-R(c-a_0)-2+\alpha,-R(d-b_0)+\beta) & P(-R(c-a_0)-1+\alpha,-R(d-b_0)-1+\beta) & -1 & 1 \\ P(-R(c-a)-1+\alpha,-R(d-b)-1+\beta) & P(-R(c-a)-2+\alpha,-R(d-b)+\beta) & 1 & 0 \\ P(-R(c-a)-2+\alpha,-R(d-b)+\beta) & P(-R(c-a)-1+\alpha,-R(d-b)-1+\beta) & -1 & 1 \end{bmatrix}\right|$$

To compute what we want, we need the first two terms in the $R \to \infty$ asymptotics of $P(-Ru-1+\alpha, -Rv-1+\beta)$.

Using Laplace's method for asymptotics of integrals, for $3|u-v$ we get:

$$P(-Ru-1+\alpha,-Rv-1+\beta)=-\frac{i}{2\pi}\left[\frac{\zeta^{\alpha-\beta}}{u\zeta-v/\zeta}-\frac{\zeta^{\beta-\alpha}}{u/\zeta-v\zeta}\right]\frac{1}{R}$$

$$-\frac{i}{2\pi}\left\{\zeta^{\alpha-\beta}\left[\frac{\alpha\zeta-\beta/\zeta}{(u\zeta-v/\zeta)^2}-\frac{u/\zeta+v\zeta}{(u\zeta-v/\zeta)^3}\right]-\zeta^{\beta-\alpha}\left[\frac{\alpha/\zeta-\beta\zeta}{(u/\zeta-v\zeta)^2}-\frac{u\zeta+v/\zeta}{(u/\zeta-v\zeta)^3}\right]\right\}\frac{1}{R^2}$$

$$+O\left(\frac{1}{R^3}\right),\quad R\to\infty.$$

Know from Theorem 1 that

$$\omega(E(Ra_0,Rb_0),E(Ra,Rb),W(Rc,Rd))\sim c\frac{1}{R^2}$$

Also

$$\omega(E(Ra_0+\alpha,Rb_0+\alpha),E(Ra,Rb),W(Rc,Rd))\sim c\frac{1}{R^2},$$

because they shrink to same points in fine mesh limit.

So to determine the asymptotics of their difference, we need the subdominant term in $\omega(E(Ra_0,Rb_0),E(Ra,Rb),W(Rc,Rd))$ — term in $\frac{1}{R^3}$. This is why we need terms in $1/R^2$ in asymptotics of P.

$T_{\alpha,\beta}$ comes out to be

$$\frac{p(\alpha, \beta, a_0, b_0, a, b, c, d)}{[(a_0 - a)^2 + (a_0 - a)(b_0 - b) + (b_0 - b)^2][(a_0 - c)^2 + (a_0 - c)(b_0 - d) + (b_0 - d)^2]} \frac{1}{R}$$

$$+ O(R^{-2}),$$

where $p(\alpha, \beta, a_0, b_0, a, b, c, d)$ is an irreducible homogenous polynomial of degree 4 in the 8 variables, having 76 terms.

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where $p(\alpha,\beta,a_0,b_0,a,b,c,d)$ is an irreducible homogenous polynomial of degree 4 in the 8 variables, having 76 terms.

Time to remember our physical analogies!

Superposition

Using the above approach one readily finds that

$$\frac{\omega(E(Ra_0+\alpha, Rb_0+\beta), E(a,b))}{\omega(E(Ra_0, Rb_0), E(a,b))} - 1 =$$

$$\frac{\alpha(2(a_0-a)+b_0-b)+\beta(a_0-a+2(b_0-b))}{(a_0-a)^2+(a_0-a)(b_0-b)+(b_0-b)^2}\frac{1}{R} + O(R^{-2})$$

and

$$\frac{\omega(E(Ra_0+\alpha, Rb_0+\beta), W(c,d))}{\omega(E(Ra_0, Rb_0), W(c,d))} - 1 =$$

$$\frac{\alpha(2(a_0-c)+b_0-d)+\beta(a_0-c+2(b_0-d))}{(a_0-c)^2+(a_0-c)(b_0-d)+(b_0-d)^2}\frac{1}{R} + O(R^{-2})$$

And indeed a Maple calculation readily confirms that the complicated expression from before is just the sum of the two fractions above!

Same set-up as before:

- $x_1^{(R)}, \dots, x_n^{(R)}, y_1^{(R)}, \dots, y_n^{(R)} \in \mathbb{Z}$, $\lim_{R\to\infty} \dfrac{x_i^{(R)}}{R} = x_i$, $\lim_{R\to\infty} \dfrac{y_i^{(R)}}{R} = y_i$

$$T_{\alpha,\beta}^{O_1^{(R)}}(O_1^{(R)}, \dots, O_n^{(R)}) := \frac{\omega(O_1^{(R)} + (\alpha, \beta), \dots, O_n^{(R)})}{\omega(O_1^{(R)}, \dots, O_n^{(R)})} - 1$$

Theorem 2 (C., 2020) Suppose O_i is either of type $\triangleright_k$ or of type $\triangleleft_k$, with k even, for $i = 1, \dots, n$. Then if $O_i^{(R)} = O_i(x_i^{(R)}, y_i^{(R)})$, for any integers α and β we have

$$\frac{1}{\mathrm{q}(O_1)} \frac{1}{\sqrt{\alpha^2 + \alpha\beta + \beta^2}} T_{\alpha,\beta}^{O_1^{(R)}}(O_1^{(R)}, \dots, O_n^{(R)})$$

$$= \mathrm{proj}_{(\alpha,\beta)} \frac{1}{2} \sum_{j=2}^{n} \mathrm{q}(O_j) \mathbf{E}(a_j, b_j; a_0, b_0) \frac{1}{R} + O(1/R^2), \quad R \to \infty.$$

$\mathbf{E}(a, b; c, d) := \dfrac{(c-a, d-b)}{(c-a)^2 + (c-a)(d-b) + (d-b)^2}$ vector pointing from (a, b) to (c, d) and having length $\dfrac{1}{\mathrm{d}((a, b), (c, d))}$ (d is the Euclidean distance)

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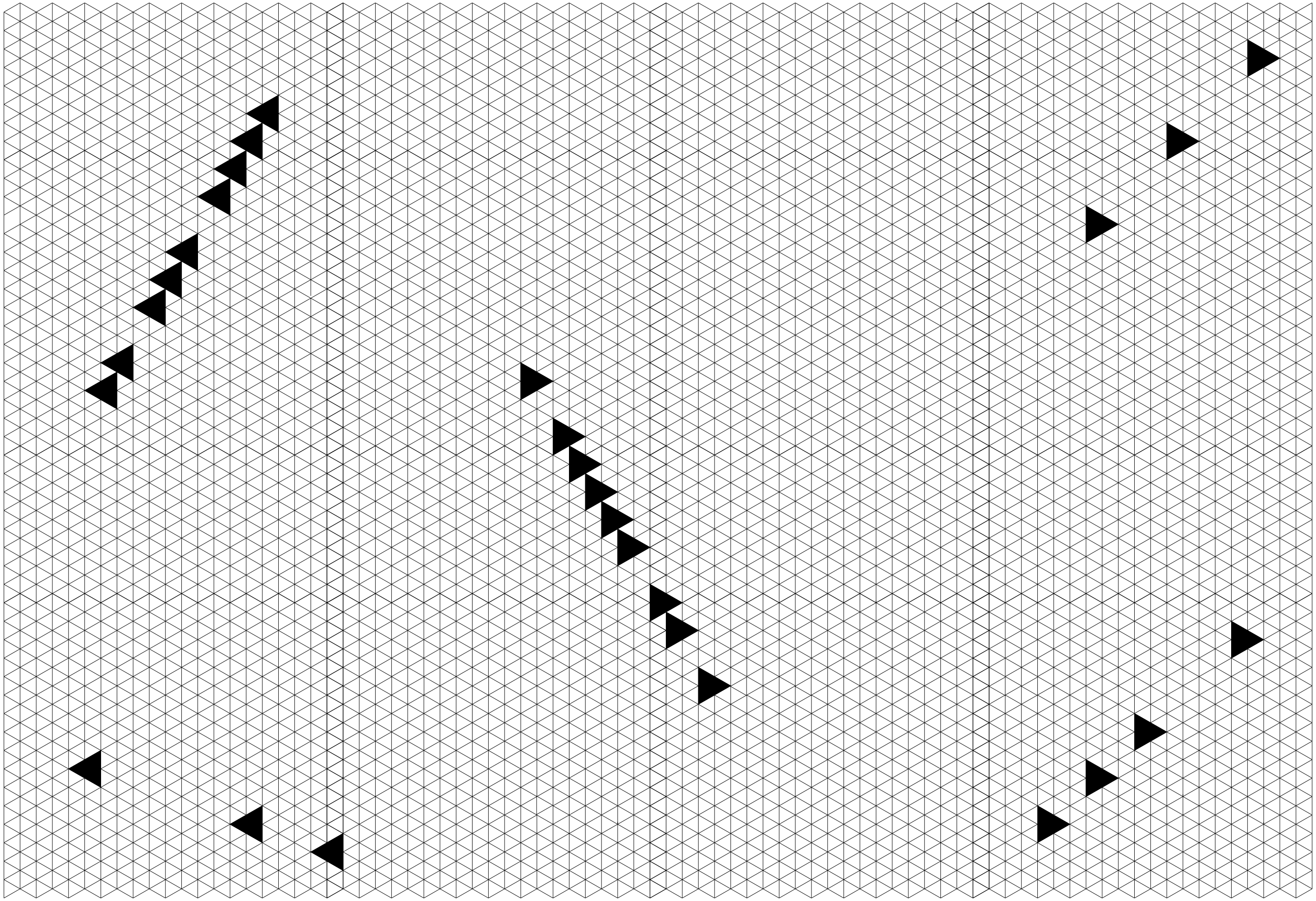
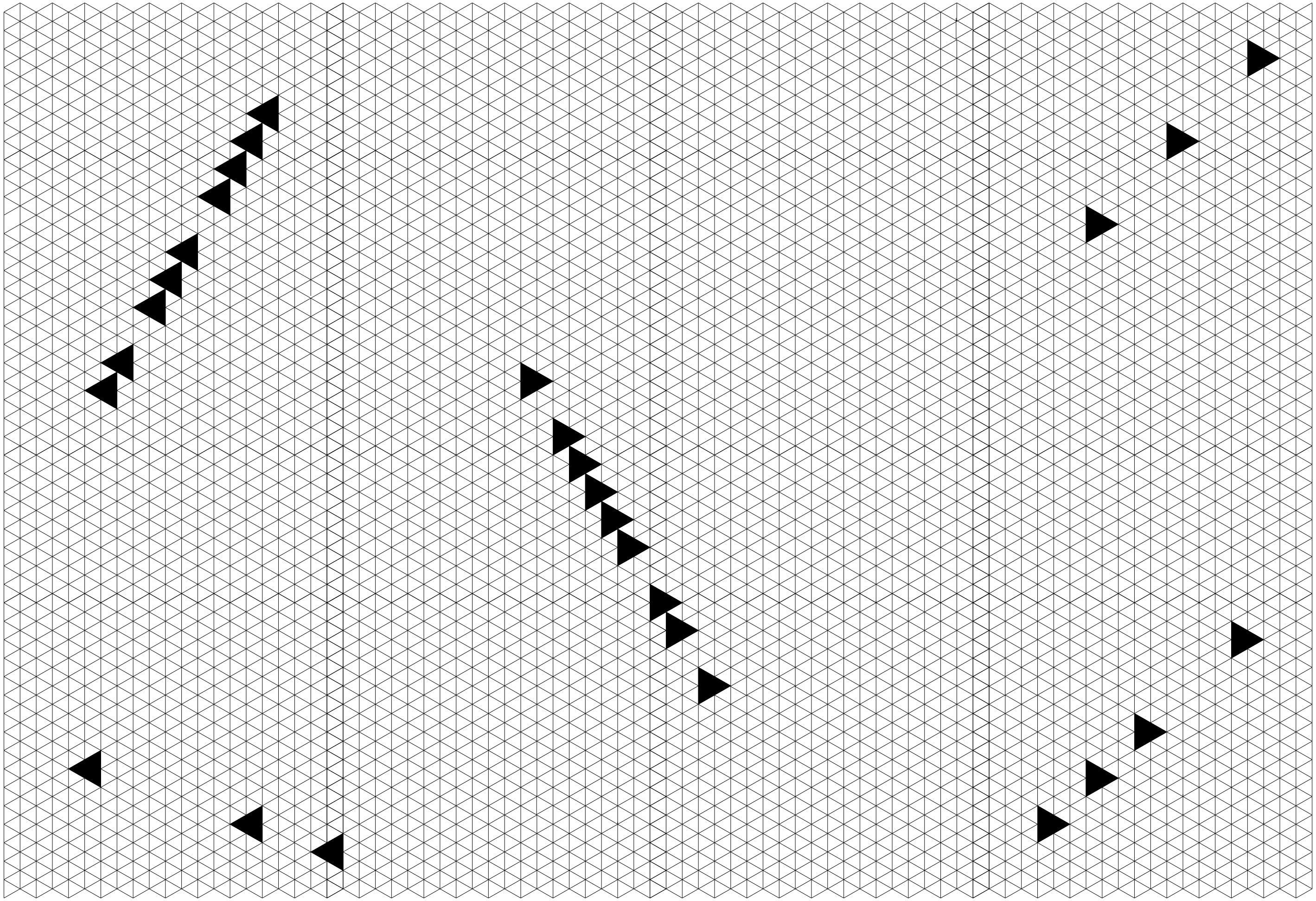
Long range — relative change is $\sim \dfrac{1}{R}$



- Statements holds more generally for “multiholes” like this



- Statement holds more generally for “multiholes” like this

- Conjecture: It holds when O_i's are arbitrary unions of unit triangles.

There is another natural field we considered before: the field **F** of average dimer orientations. Its scaling limit is also the electric field.

Main differences between F and the new field T

• Definition of **T** involves an infinitesimal, definition of **F** doesn't

• The field **F** at a point arises simply by presence of holes in their places; for **T** we need to create a hole where we want to measure it

• **T** captures in some sense "resistance" encountered as a hole is pushed through the sea of dimers

• **T** and **F** are different in the presence of boundary (C. and Krattenthaler (2011))