Cluster Algebra, Dimers and Beyond

(P. Di Francesco)

- Statistical physics in 2D

  Dimer models → free fermions (Ising)

  Vertex/face models → Conformal field theory (Potts...)

- Cluster algebra = allows to look at statistical models in a completely different manner

  mutation = change the model locally while preserving properties.

- Continuum/Thermodynamic limit: limit shapes

PLAN: (2 lectures)

- Statistical Physics
- T-system (octahedron eqn) and Cluster Algebra
- Toy case: the A_1 T-system
  solution via flat connection / Networks / Dimers
- General case: the A_∞ T-system
  solution / Networks / Domino Tilings of the Aztec diamond
- Limit shapes I: arctic circle from cluster algebra
  - arctic circle
  - generalization to periodic weights
- Limit shapes II: arctic circle from the tangent method
  - arctic circle
  - beyond dimers: osculating paths (6V, 20V)

Statistical Physics

- states = configurations
- (local) energies = weights
- Partition function $Z := \sum_{\text{states}} \prod \text{weights}$
- Thermodynamic (continuum limit)

  size $\to \infty$ mesh $\to 0$

- Critical phenomena = singularities in the thermodynamic limit

Cluster Algebra: our main example

- T-system (octahedron) relations:

$$\boxed{T_{\alpha,j,k+1}\,T_{\alpha,j,k-1} = T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k}}$$

discrete time evolution: (2+1 dim'n.)

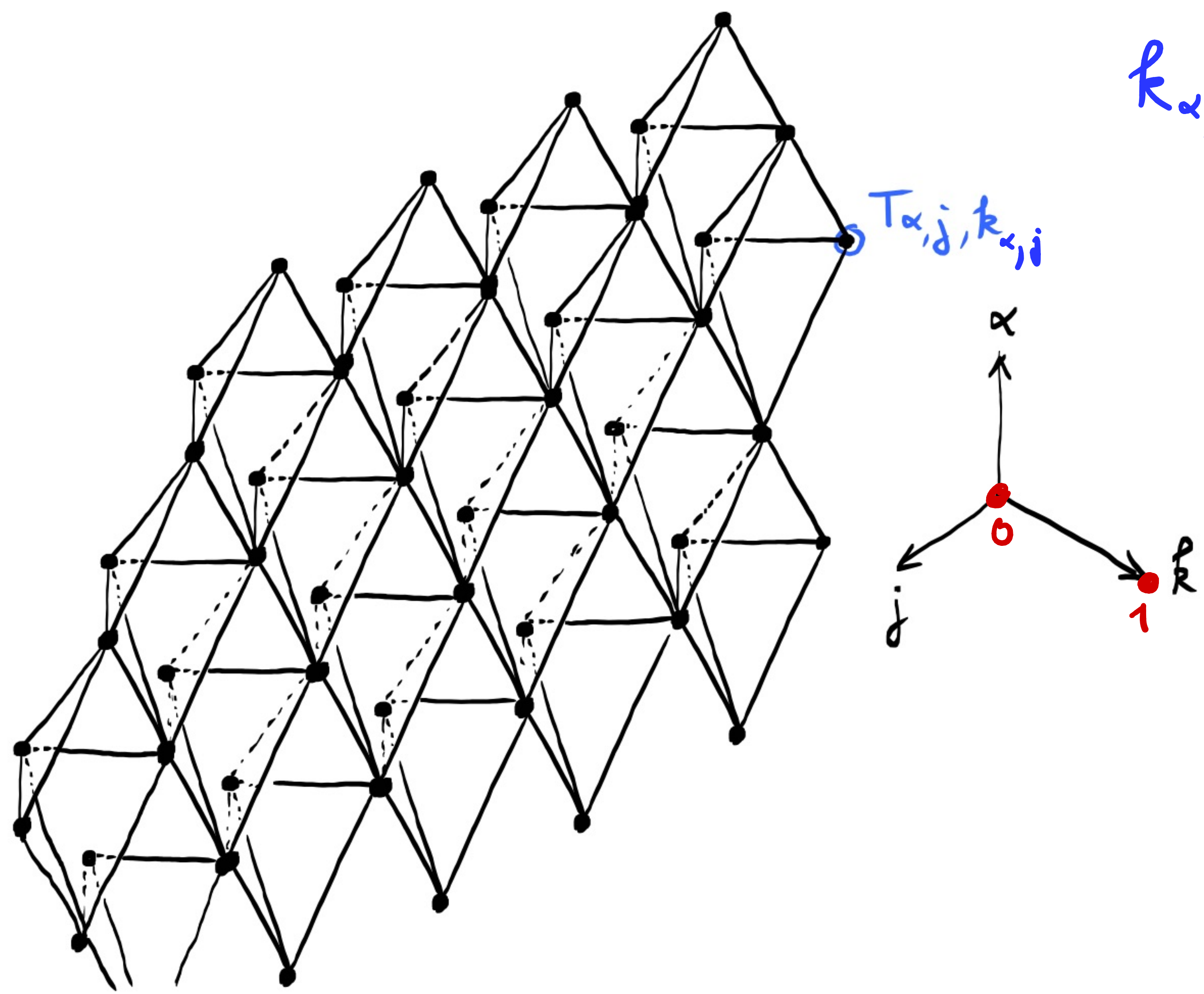
$$\begin{cases} (\alpha,j) = \text{space } (\mathbb{Z}^2) \\ k = \text{time } (\mathbb{Z}) \end{cases}$$

- initial (Cauchy) data = "stepped surface"

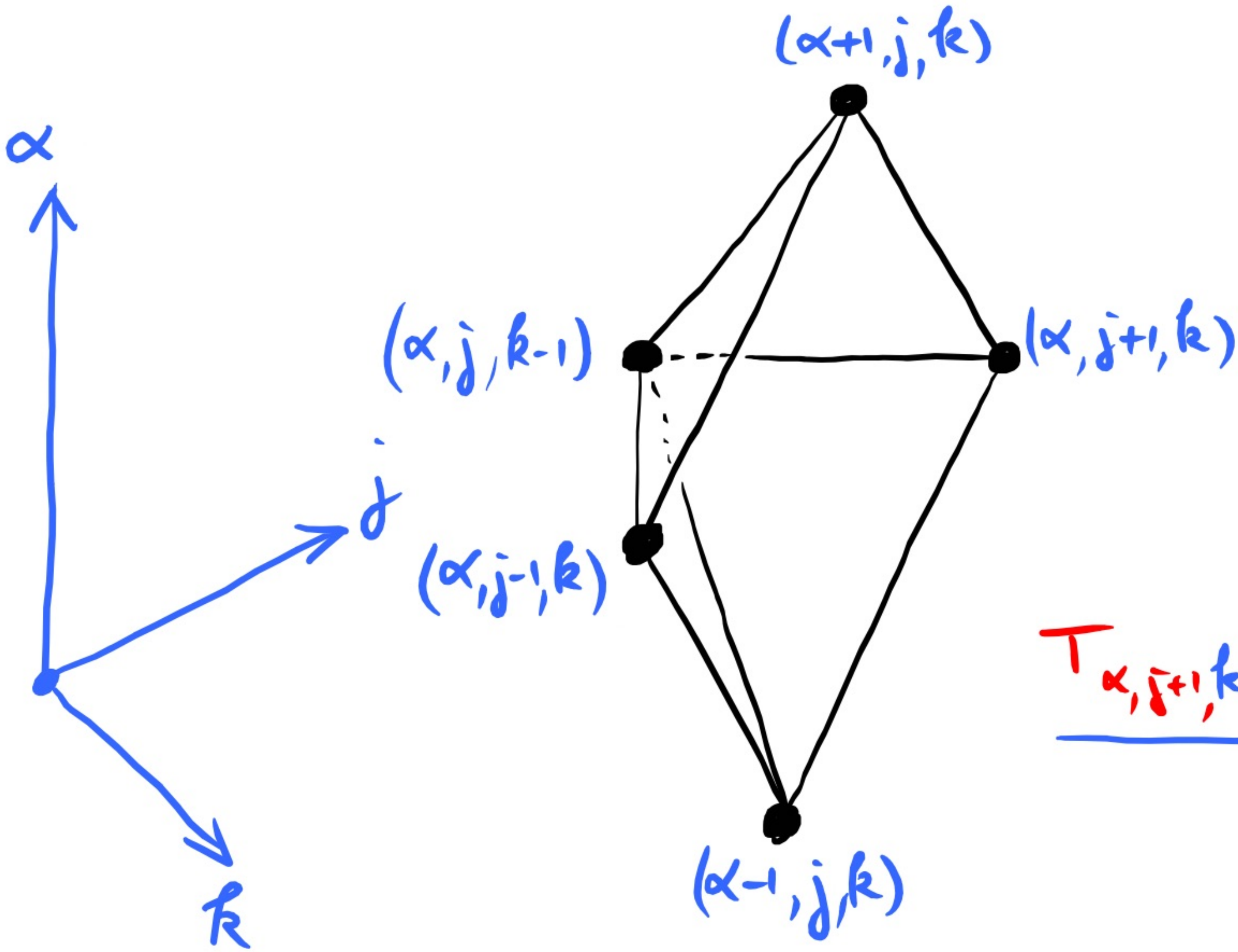
$$\left\{ T_{\alpha,j,k_{\alpha,j}} \;\Big|\; |k_{\alpha,j+1}-k_{\alpha,j}| = |k_{\alpha+1,j}-k_{\alpha,j}| = 1 \right\}$$

- Flat initial data

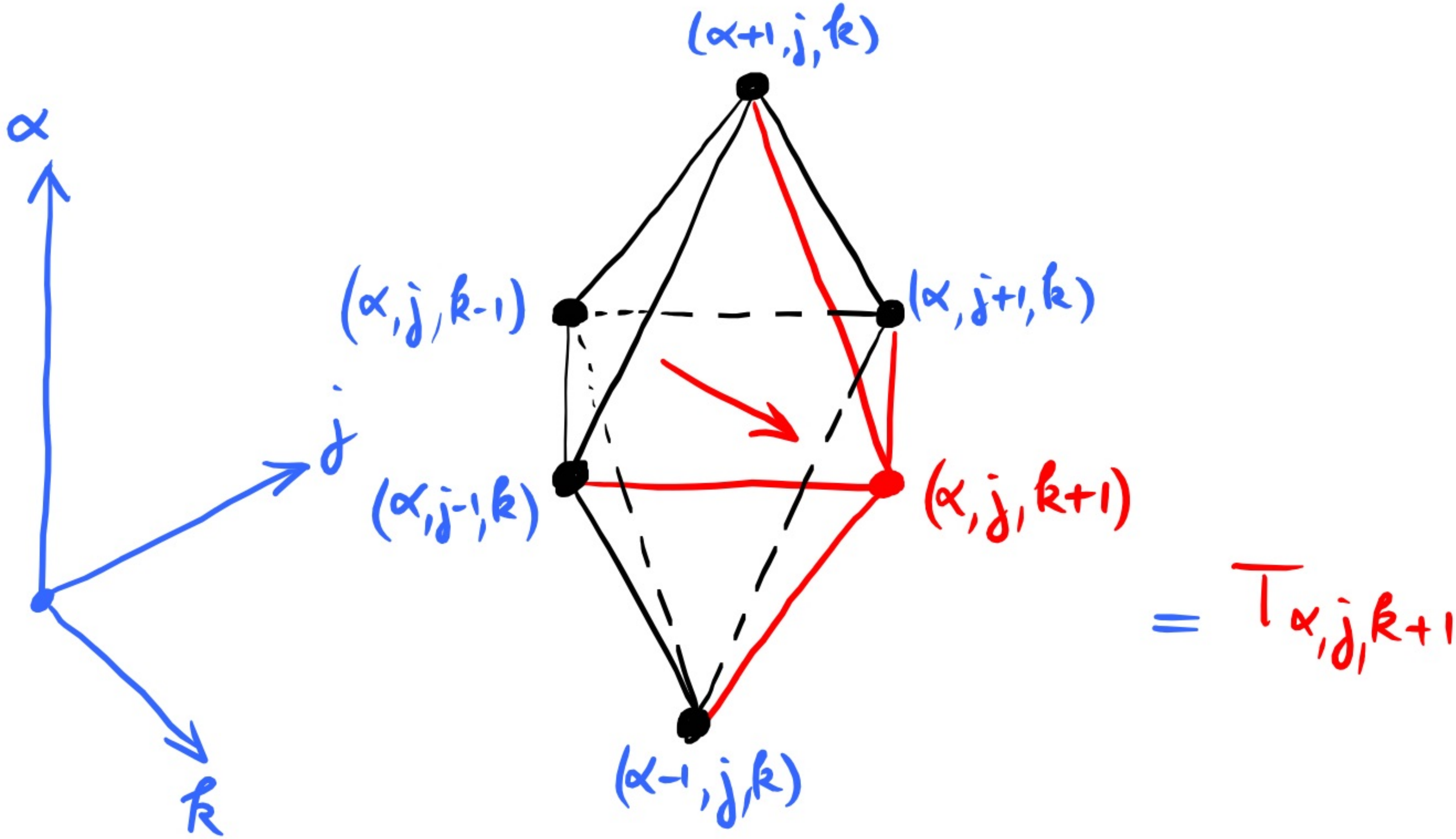
$k_{\alpha,j} \in \{0,1\}$



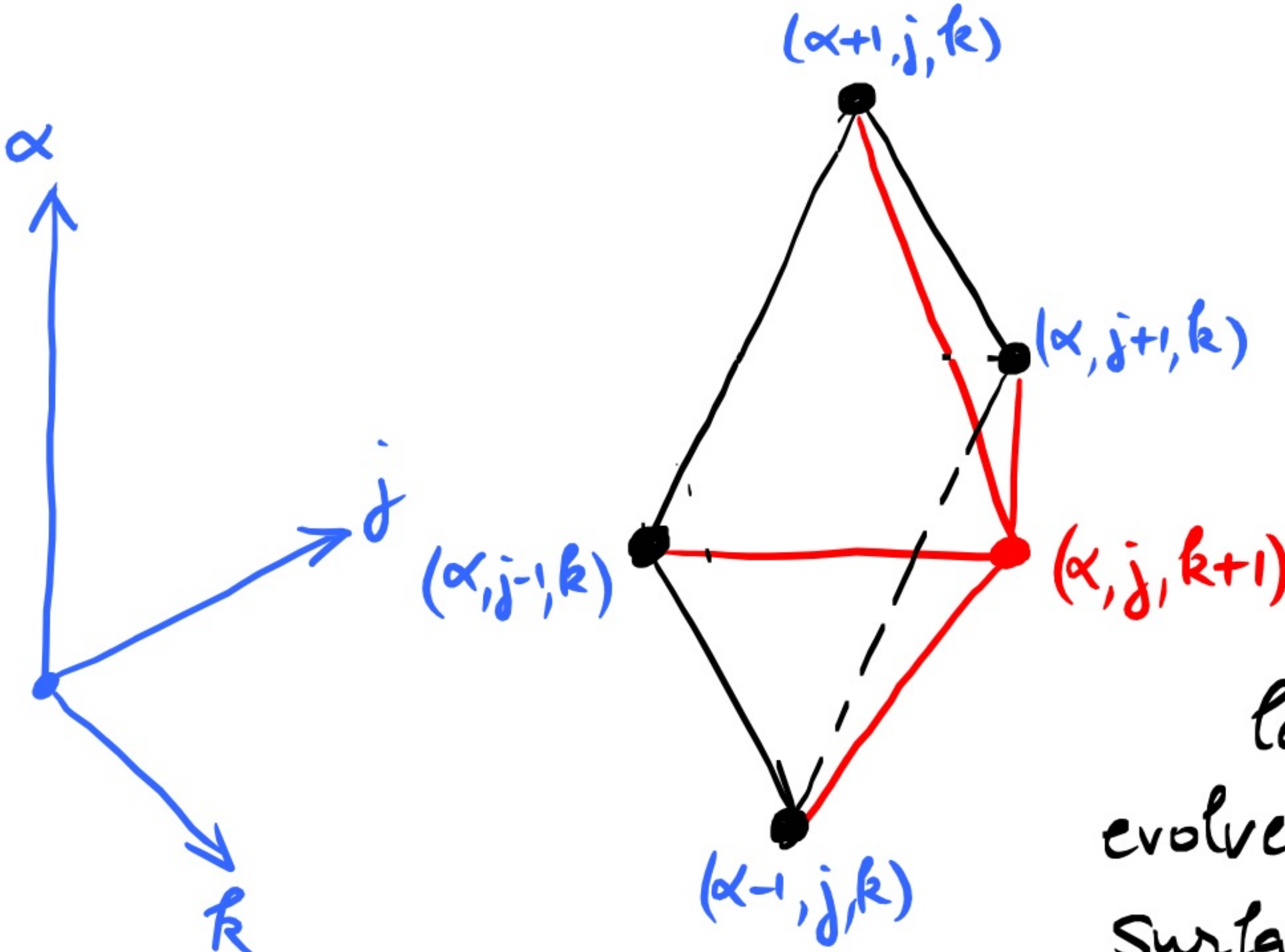
- "octahedral" evolution



$$\text{new } T = \frac{T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k}}{T_{\alpha,j,k-1}}$$



α
j
k
$(\alpha+1, j, k)$
$(\alpha, j, k-1)$
$(\alpha, j+1, k)$
$(\alpha, j-1, k)$
$(\alpha, j, k+1)$
$(\alpha-1, j, k)$
$= T_{\alpha, j, k+1}$

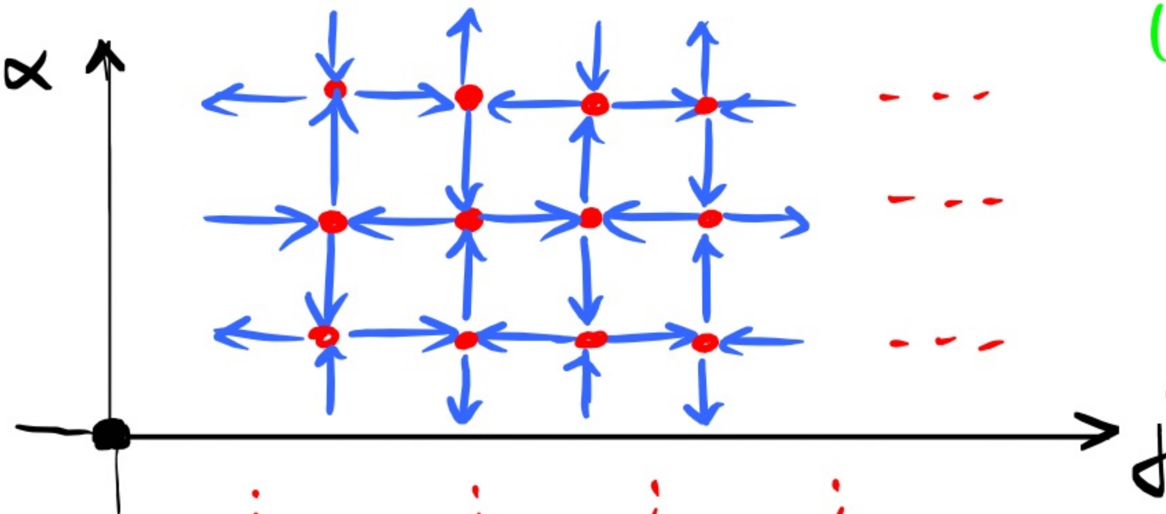


local move that evolves the stepped surface by "adding" an octahedron

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:



Cluster :

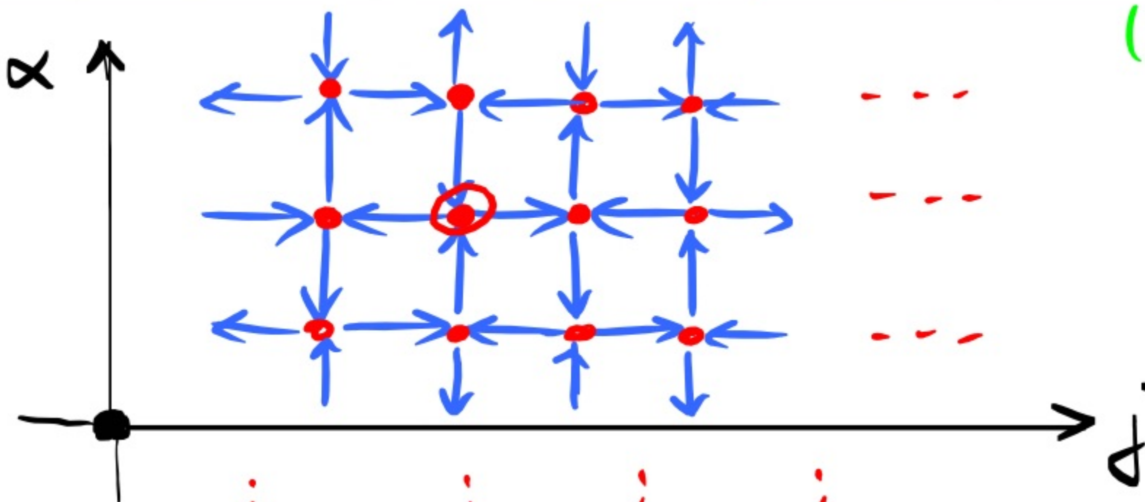
$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & T_{1-10} & T_{101} & T_{110} & T_{121} & \cdots \\
\cdots & T_{0-11} & T_{000} & T_{011} & T_{020} & \cdots \\
\cdots & T_{-1-10} & T_{-101} & T_{-110} & T_{-121} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:

mutation



Cluster :

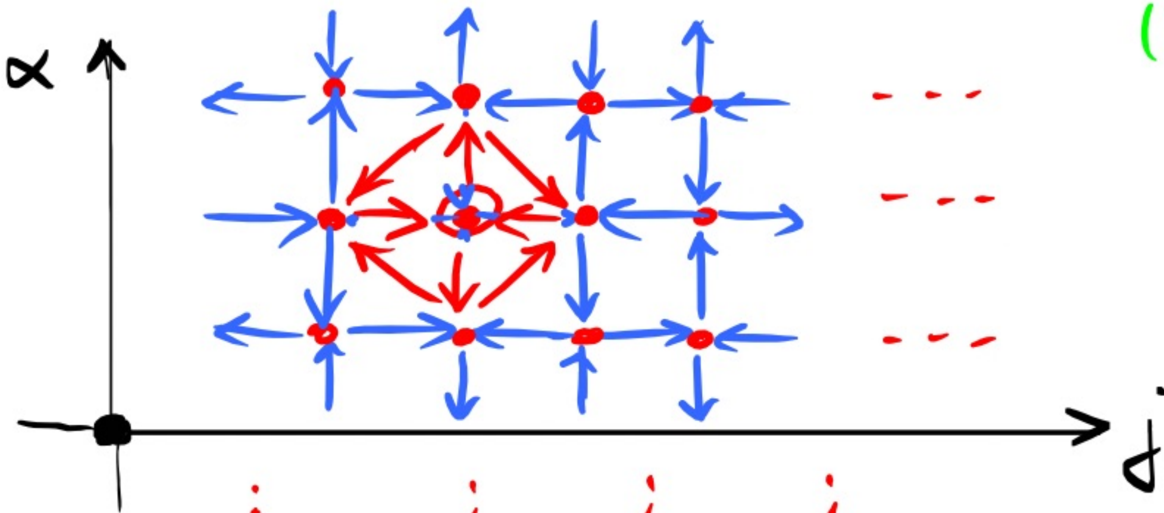
$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & T_{1-10} & T_{101} & T_{110} & T_{121} & \cdots \\
\cdots & T_{0-11} & T_{000} & T_{011} & T_{020} & \cdots \\
\cdots & T_{-1-10} & T_{-101} & T_{-110} & T_{-121} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:

mutation



Cluster :

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & T_{1,-1,0} & T_{1,0,1} & T_{1,1,0} & T_{1,2,1} & \cdots \\
\cdots & T_{0,-1,1} & \bigcirc & T_{0,1,1} & T_{0,2,0} & \cdots \\
\cdots & T_{-1,-1,0} & T_{-1,0,1} & T_{-1,1,0} & T_{-1,2,1} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

$$\frac{T_{0,-1,1}T_{0,1,1}+T_{1,0,1}T_{-1,0,1}}{T_{000}}$$

Boundary conditions

- none "A_∞ T-system"
- half-space "$A_{\infty/2}$ T-system" $\alpha \geq 0$

(above a plane) $(T_{0,j,k} = 1)$

- strip "A_r T-system $0 \leq \alpha \leq r+1$

(between 2 planes) $(T_{0,j,k} = T_{r+1,j,k} = 1)$

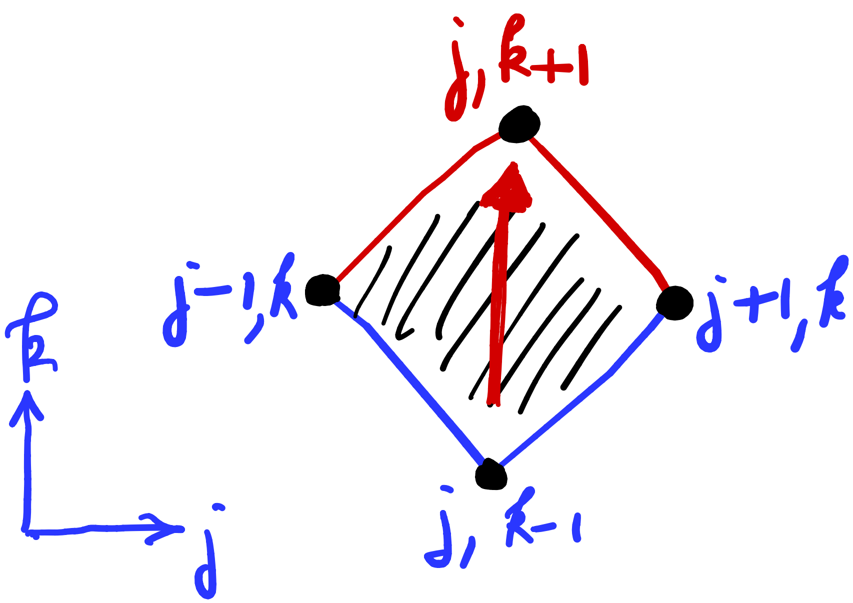
Toy case: A_1 T-system

- 1 layer between the planes $\alpha=0$ and $\alpha=2$

$$T_{0,j,k} = T_{2,j,k} = 1 \qquad T_{1,j,k} = T_{j,k}$$

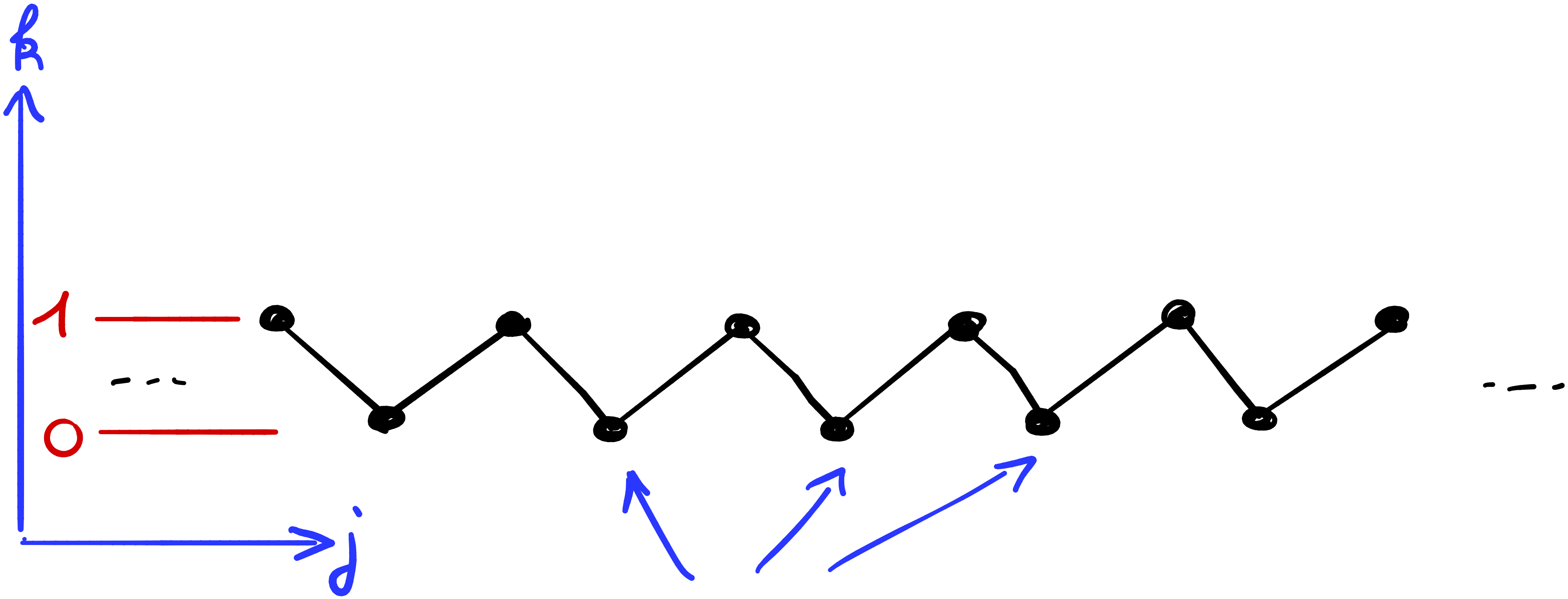
- octahedron relation becomes (1+1 dim'n)

$$T_{j,k+1}\, T_{j,k-1} = T_{j+1,k}\, T_{j-1,k} + 1$$



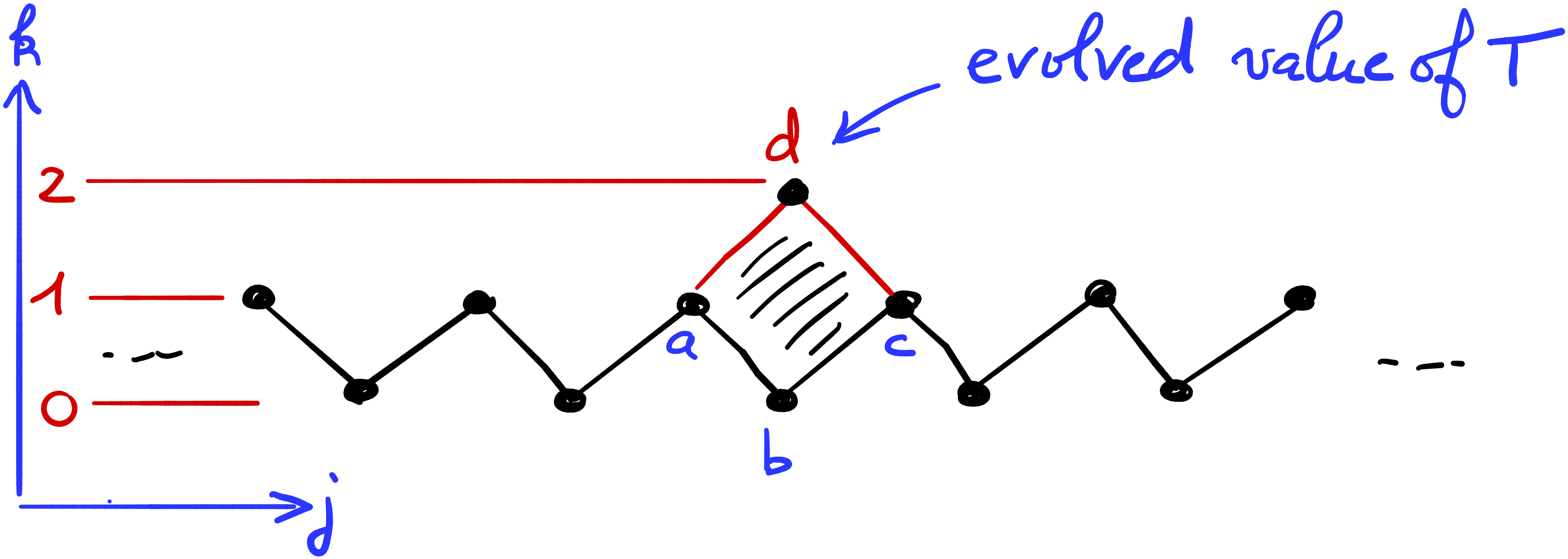
also $\begin{vmatrix} T_{j,k-1} & T_{j-1,k} \\ T_{j+1,k} & T_{j,k+1} \end{vmatrix} = 1$

- Flat initial data



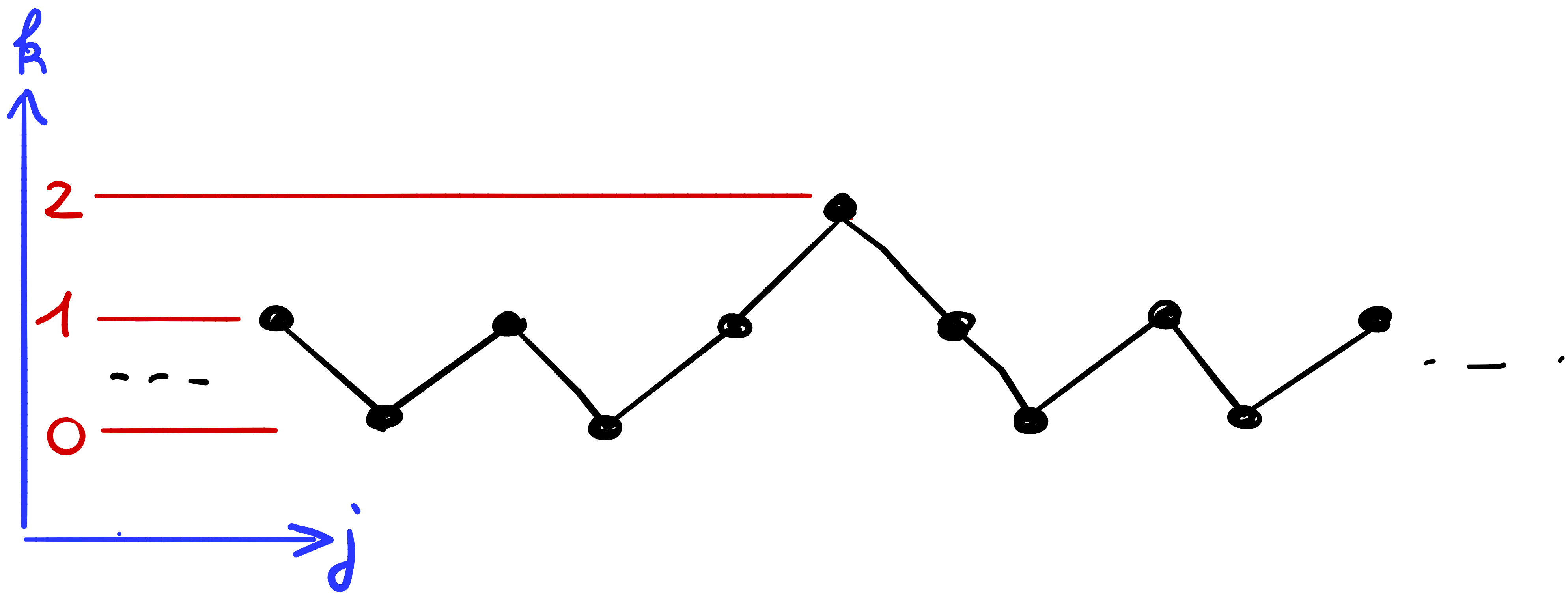
assigned initial values of T at each of these vertices.

- Evolution



$$bd = ac + 1$$

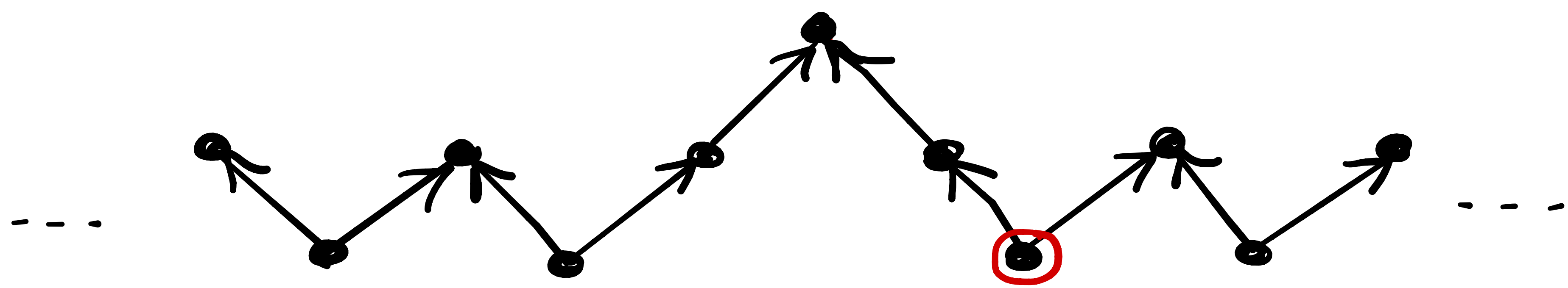
• Evolution → another admissible initial data



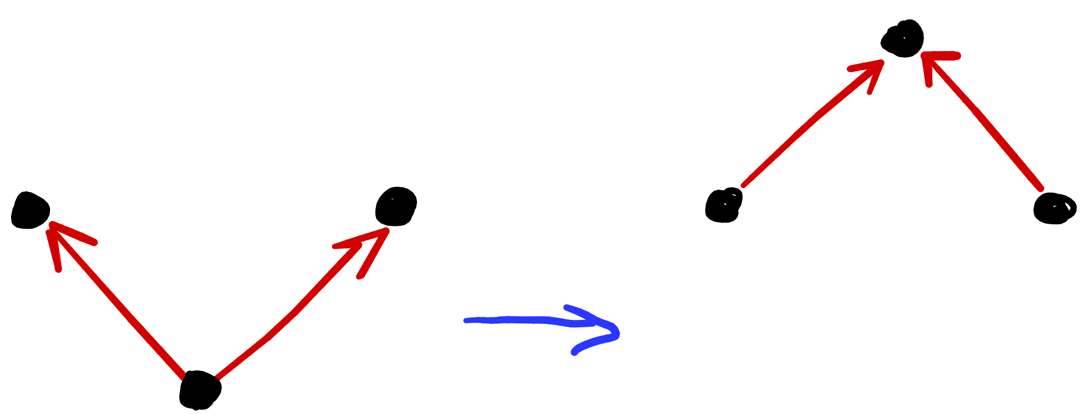
etc.

Associated Cluster algebra

- quiver = orient all edges up



compatible w/ quiver mutations



Associated Cluster algebra

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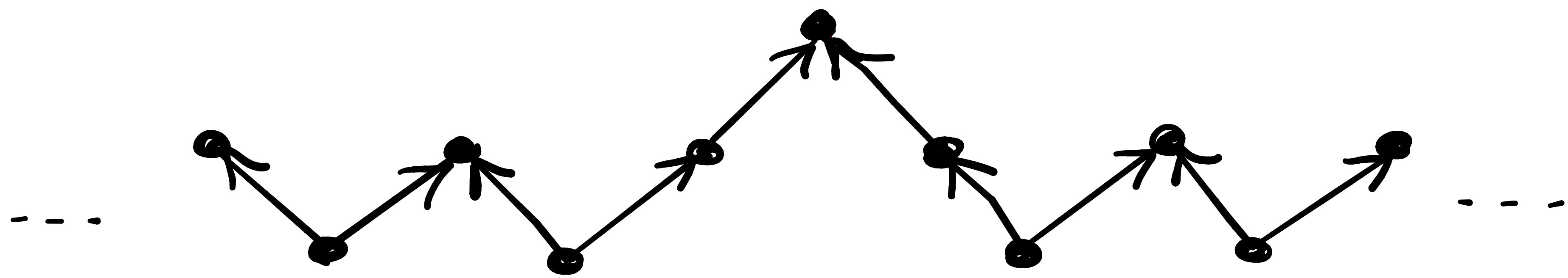
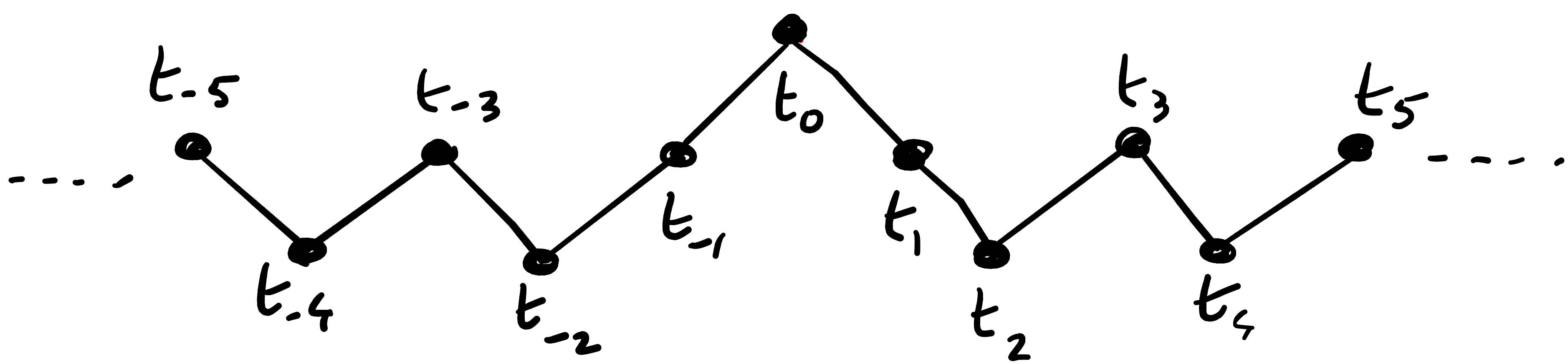
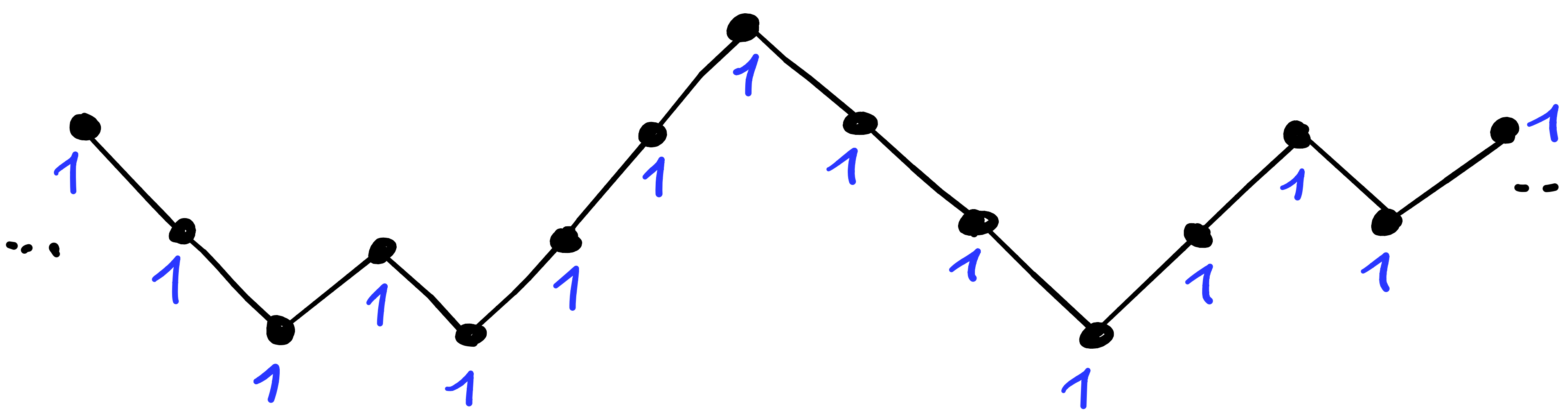


- cluster = corresponding init data



Q: express $T_{j,k}$ in terms of any given initial data

A: Flat connection (integrability).



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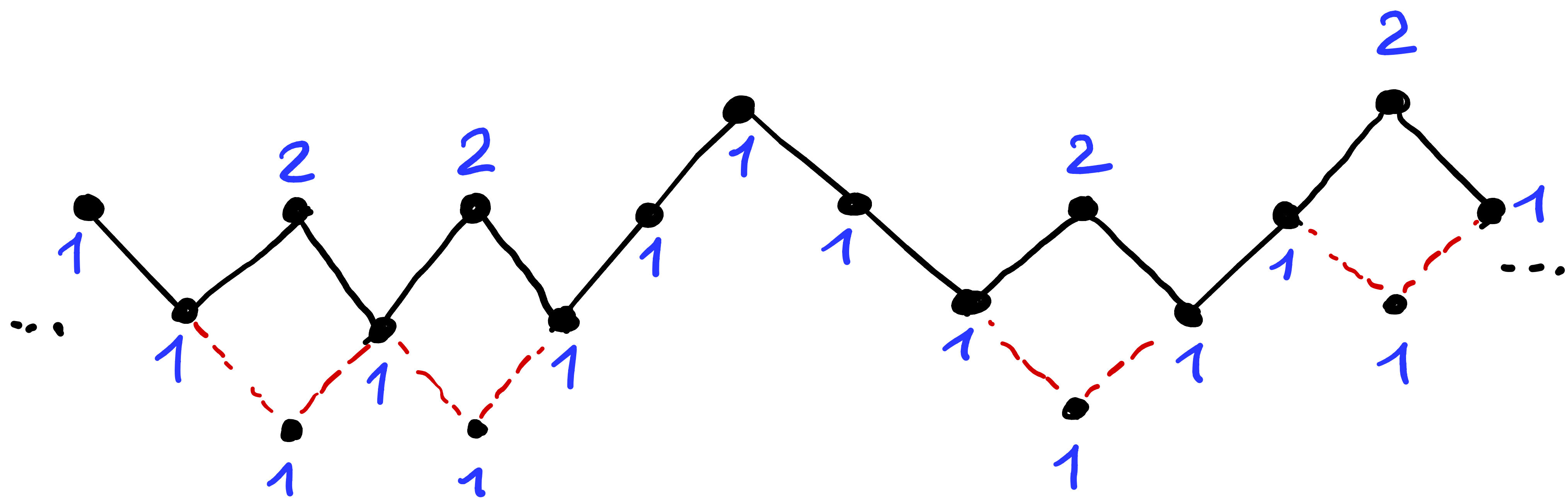

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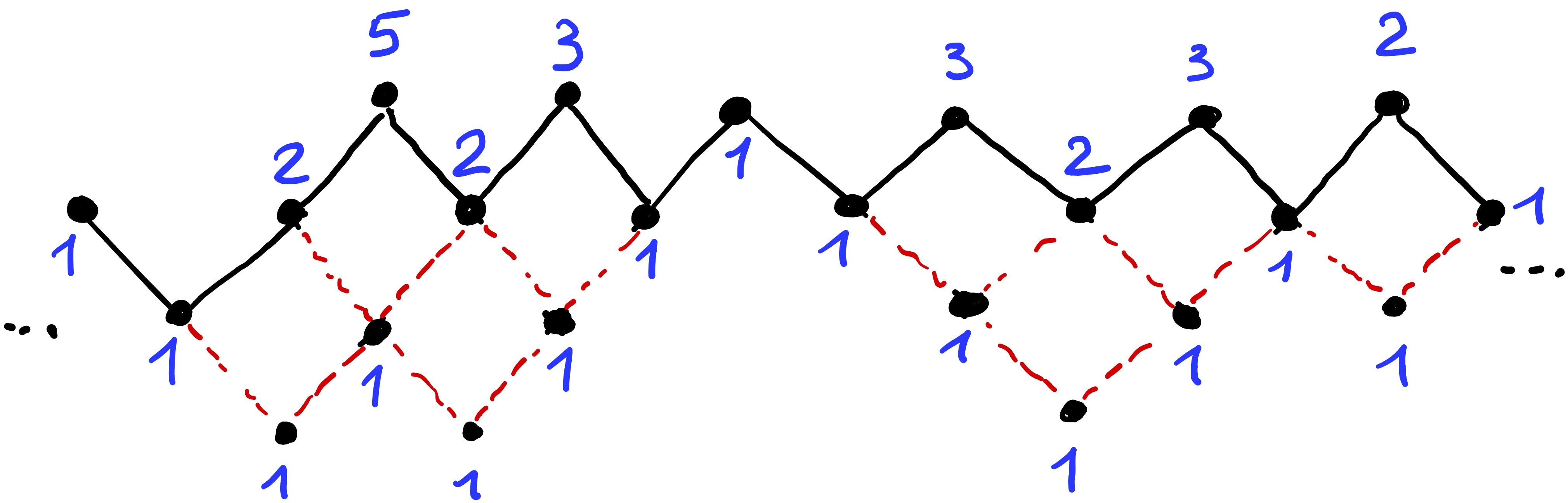
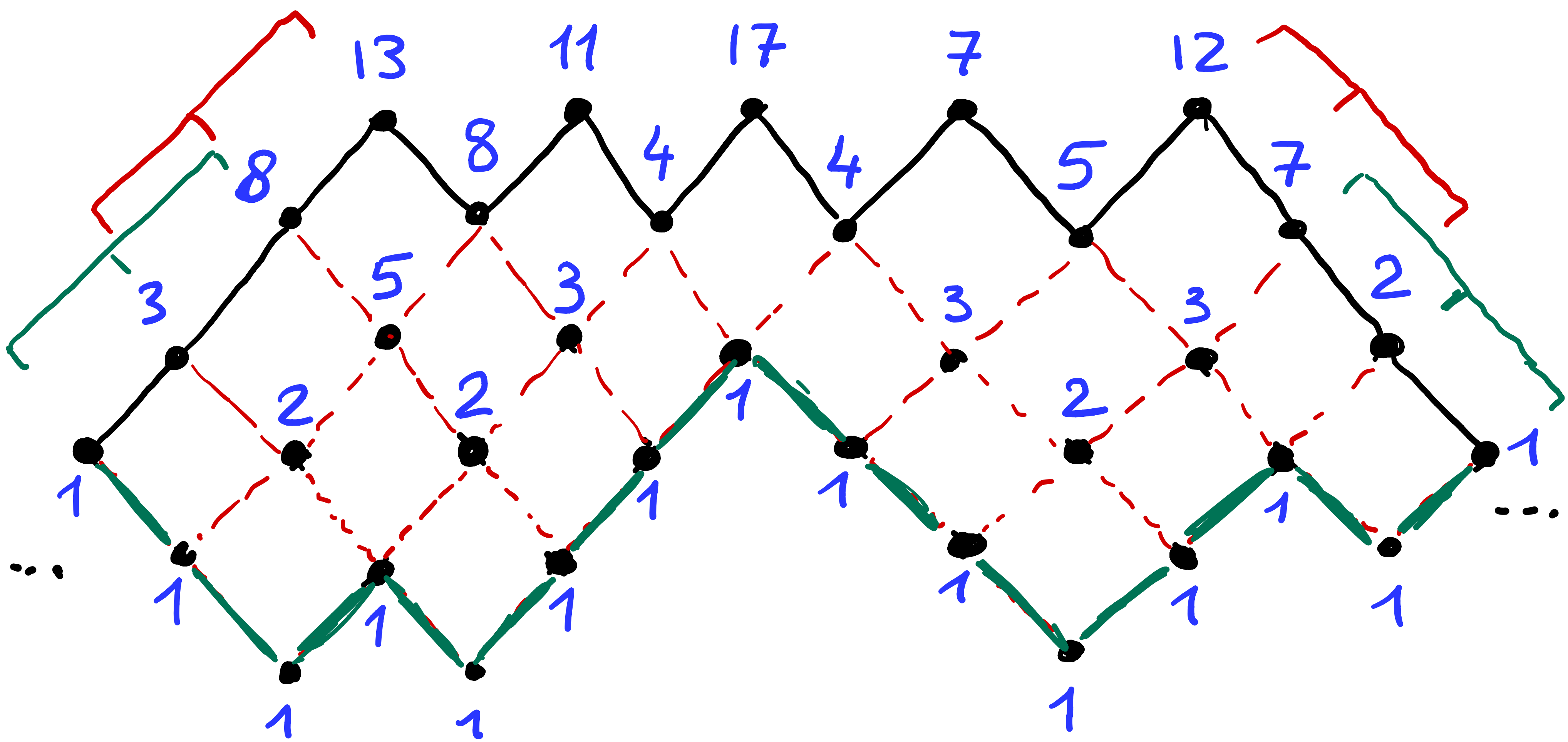


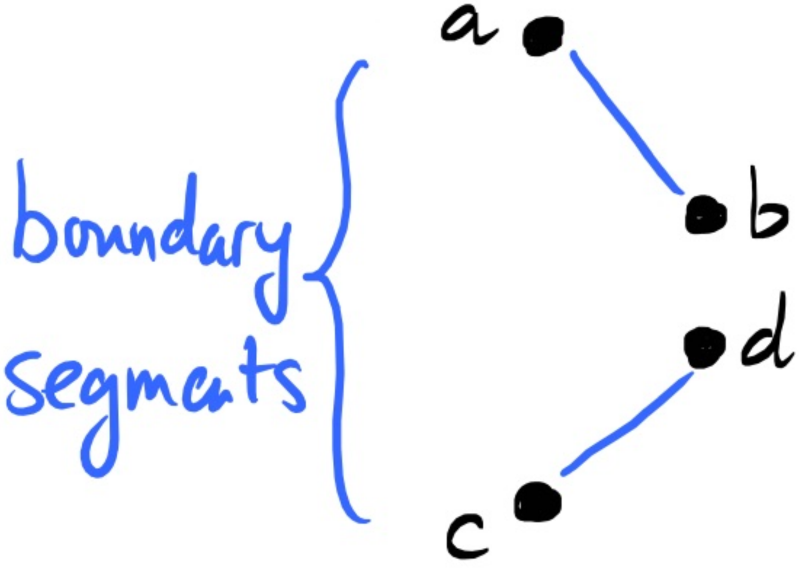
Q: express $T_{j,k}$ in terms of any given initial data

A: Flat connection (integrability).

CONSERVATION LAWS!



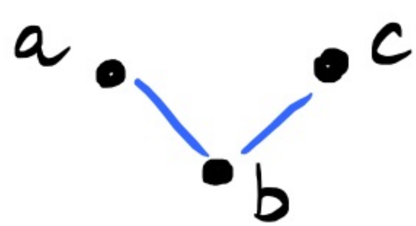
MATRIX REPRESENTATION:



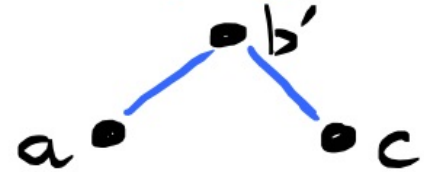
$$\longrightarrow \quad D(a,b) = \begin{pmatrix} \frac{a}{b} & \frac{1}{b} \\ 0 & 1 \end{pmatrix}$$

$$\longrightarrow \quad U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix}$$

$D(a,b)\,U(b,c)$

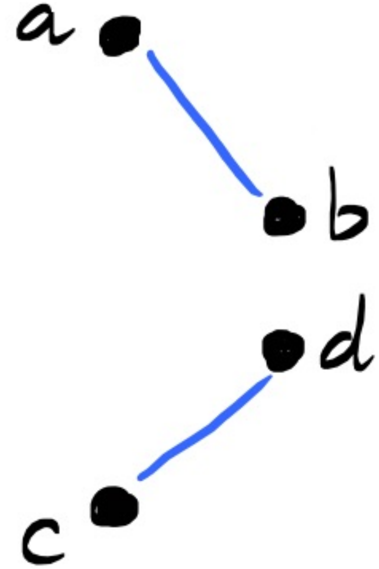


$U(a,b')\,D(b',c)$



MATRIX REPRESENTATION:

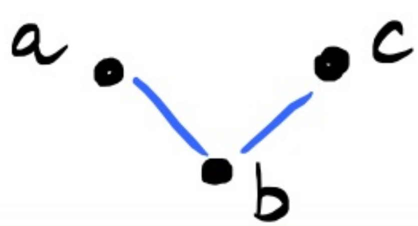
boundary segments

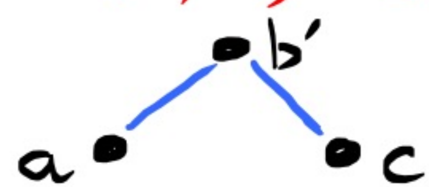


$$\rightarrow \quad D(a,b) = \begin{pmatrix} \frac{a}{b} & \frac{1}{b} \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow \quad U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix}$$

$$D(a,b)\,U(b,c) = U(a,b')\,D(b',c) \quad \Leftrightarrow$$

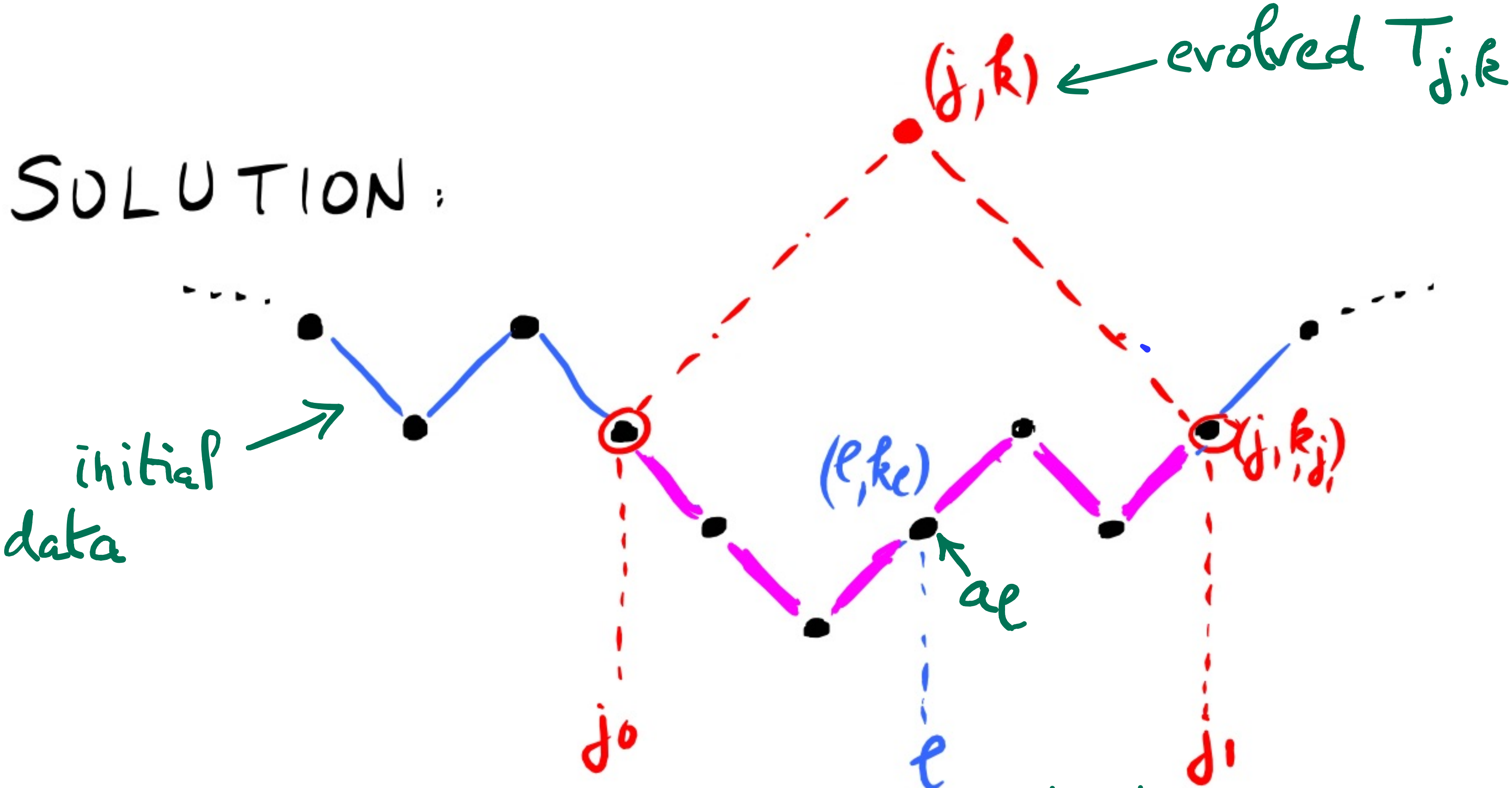




$$\boxed{bb' = ac + 1}$$

(Flat GL(2) connection) $\leftrightarrow$ (integrability)

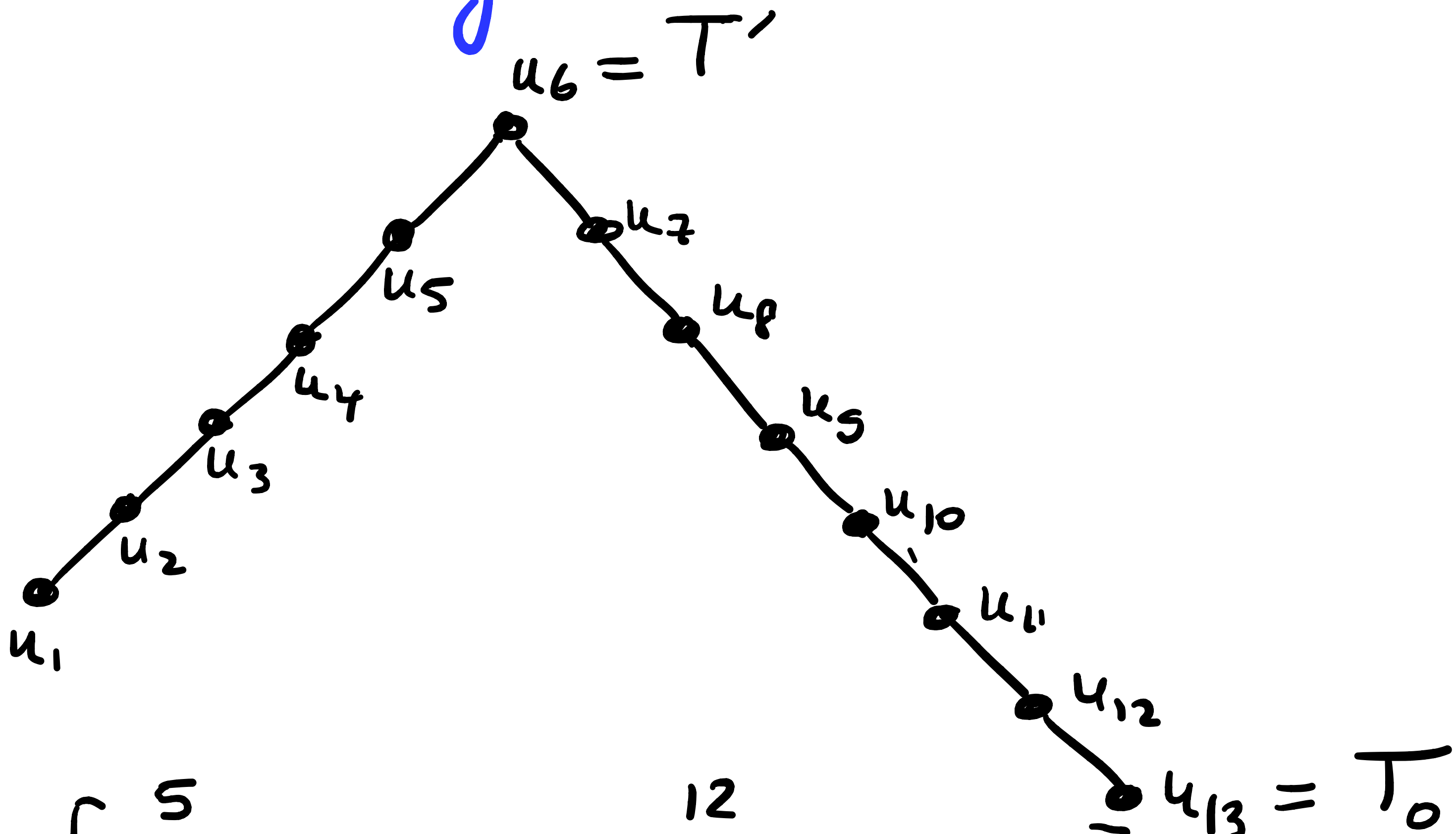
SOLUTION:



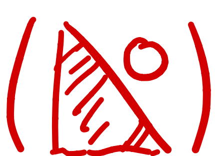
init. data

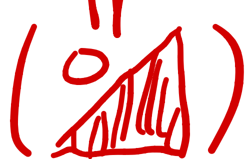
$$\frac{T_{j,k}}{T_{j_1,k_{j_1}}} = \Bigg[\underbrace{\prod_{\ell=j_0}^{j_1-1} \begin{Bmatrix} D \\ U \end{Bmatrix} (a_\ell, a_{\ell+1})}_{\text{"Tranfer matrix"}} \Bigg]_{1,1} \leftarrow (1,1)\text{ element}$$

Proof compute the relevant quantity on the "pyramid" configuration



$$\frac{T'}{T_0} = \Bigg[\underbrace{\prod_{1}^{5} U(u_i, u_{i+1})}_{\text{lower triang}} \; \underbrace{\prod_{6}^{12} D(u_i, u_{i+1})}_{\text{upper triang}} \Bigg]_{1,1} = 1 \times \frac{u_6}{u_{13}}$$





Note: (1) the arguments of D, U are values of T_{j,k_j} from the initial data

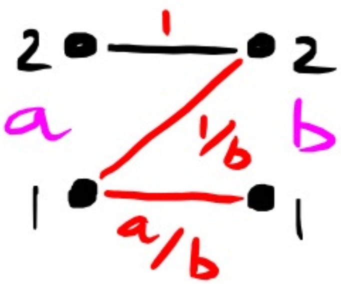
(2) entries are all >0 Laurent monomials

$\Rightarrow$ LAURENT POSITIVITY

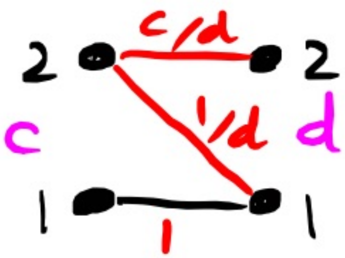
NETWORK FORMULATION

weighted graphs (oriented left→right)

$$D(a,b) = \begin{pmatrix} \frac{a}{b} & \frac{1}{b} \\ 0 & 1 \end{pmatrix} =$$



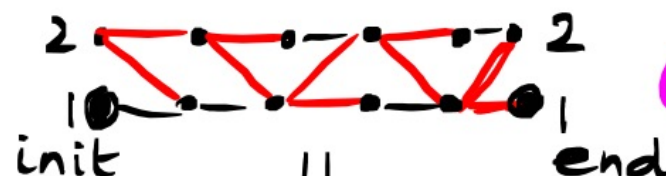
$$U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix} =$$



$$(U\ U\ D\ U\ D)_{1,1}$$

Product ↓ Concatenate

||



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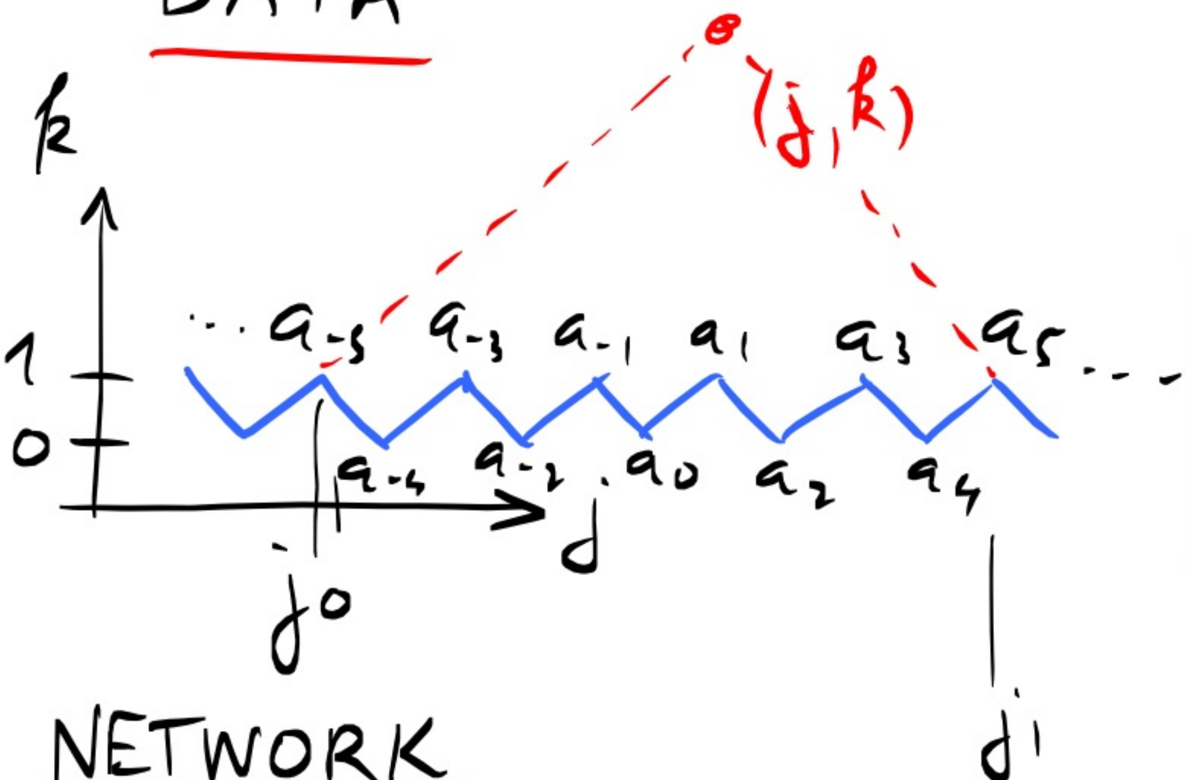
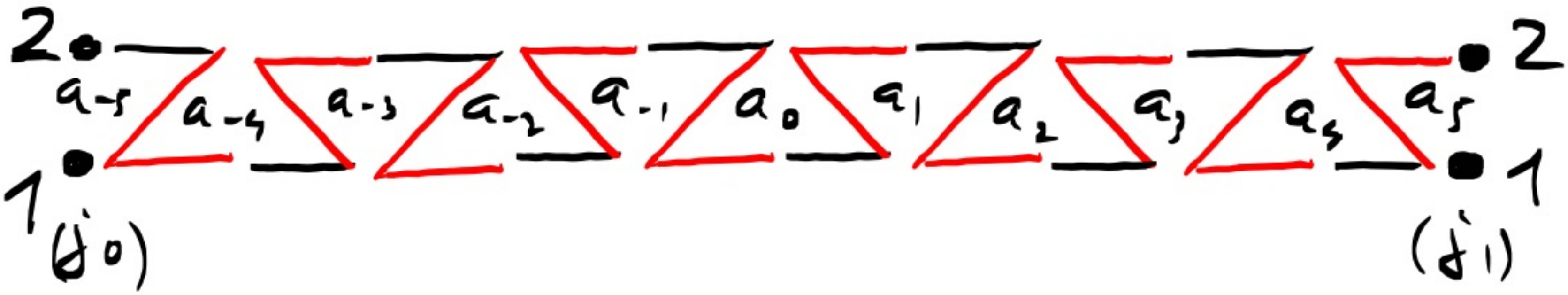
PF for paths $1 \to 1$ on the network.

PARTICULAR CASE: THE "FLAT" INITIAL DATA



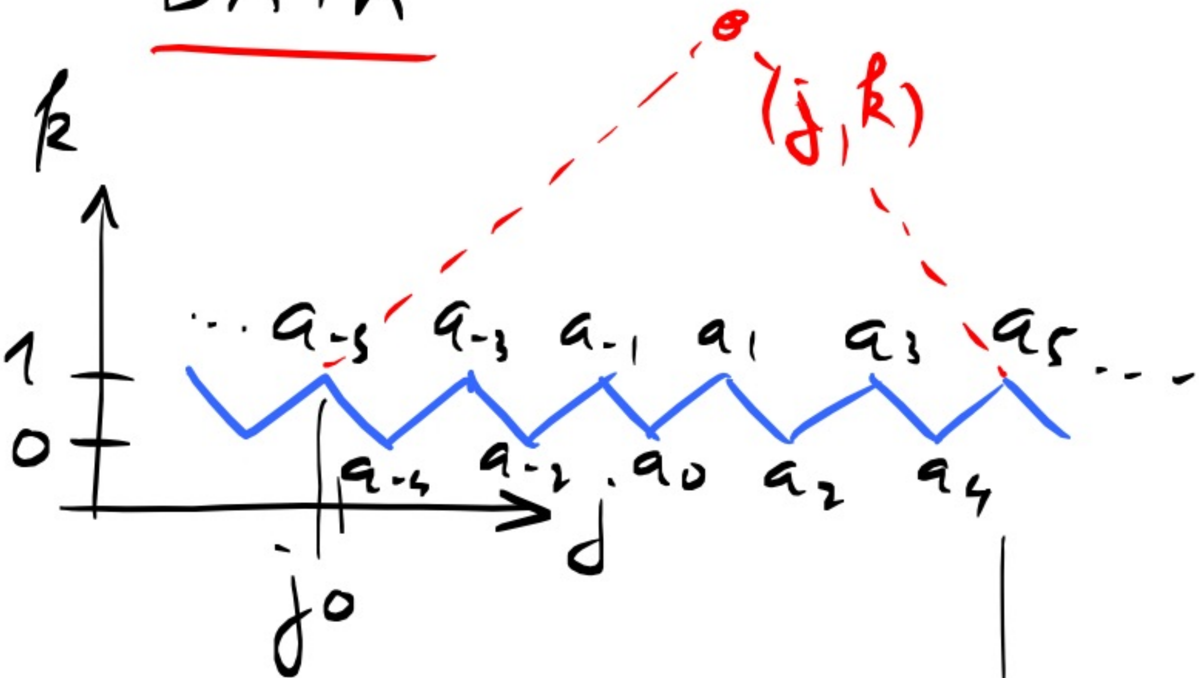
$$T_{jk} = \left(\prod_{j_0}^{j_1-1} DU\right)_{1,1} a_{j_1}$$

NETWORK



PARTICULAR CASE: THE "FLAT" INITIAL

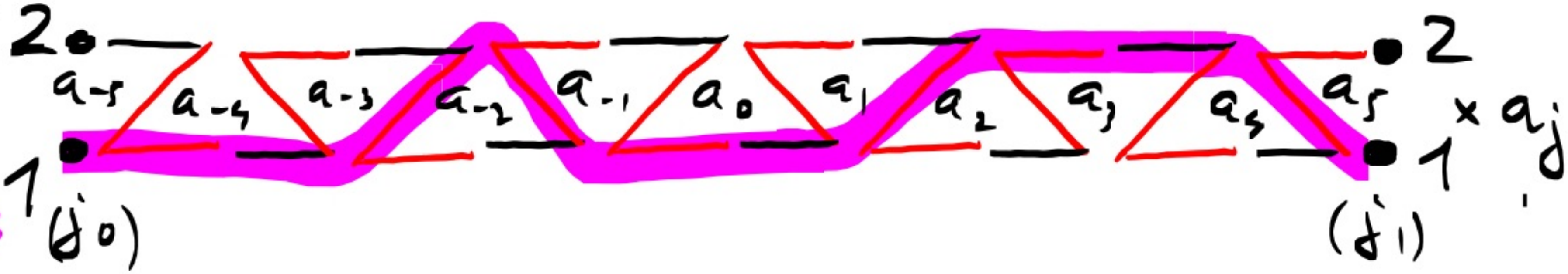
DATA



$$T_{jk} = \left(\prod_{j_0}^{j_1 - 1} DU \right)_{1,1} a_{j_1}$$

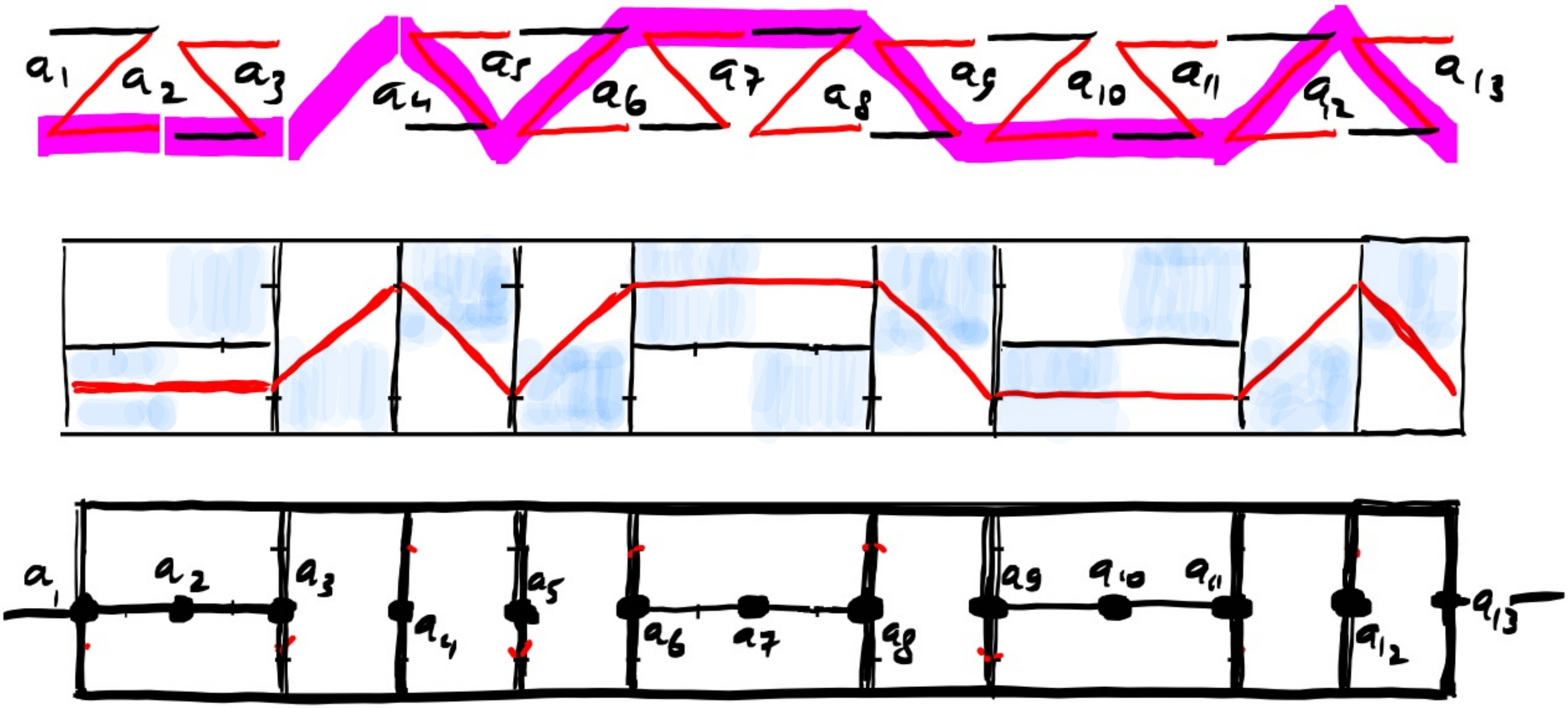
NETWORK

$T_{jk} = \sum_{\text{paths}}$



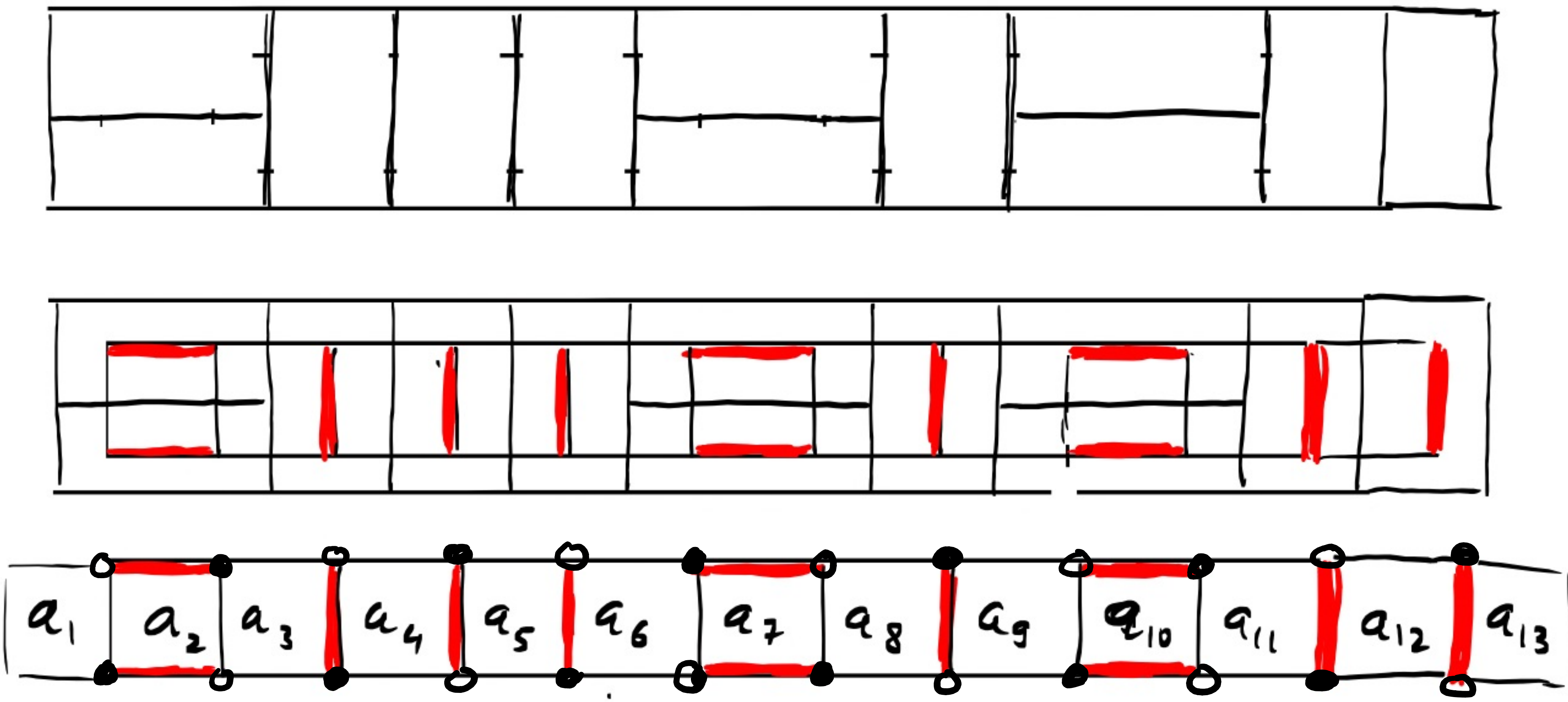
$\times a_{j_1}$

FROM PATHS TO DOMINO TILINGS



Weight = $\prod a_i^{N_i - 3}$ → degree of vertex i

From Dominos to Dimers



$$\text{Weight} = \prod a_i^{1-N_i}$$

dimers on square i

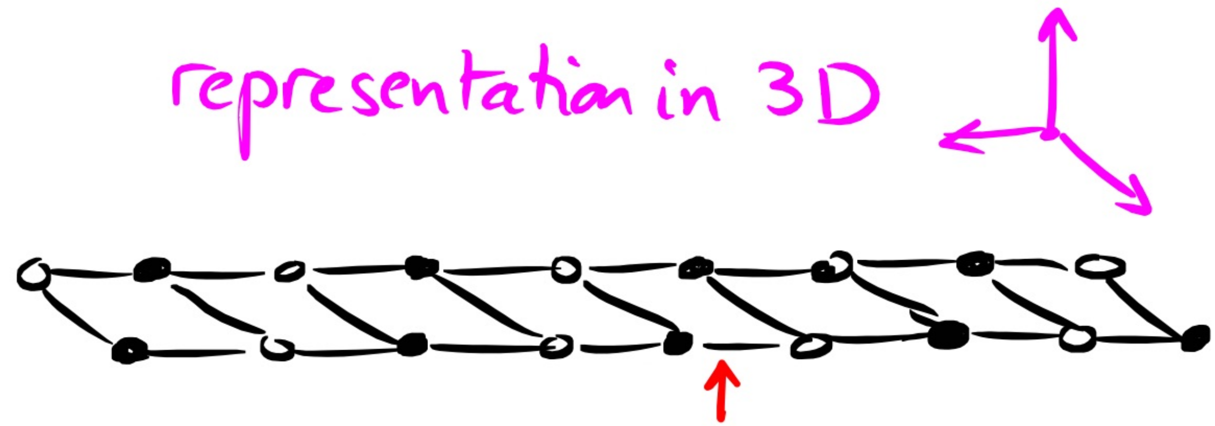
$(N_i \in \{0,1,2\})$

CONCLUSION:

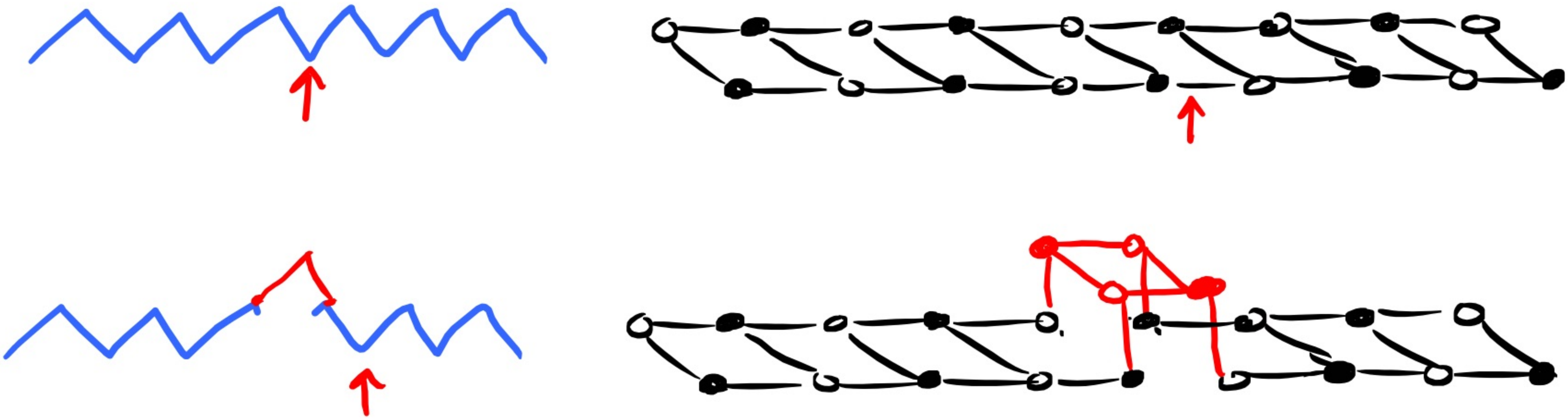
- We can think of D, U as local transfer matrices for tiling/dimer model.
- Laurent positivity $\Longleftrightarrow$ positivity of the Boltzmann weights of the statistical model describing the cluster variables for given initial data.

MUTATIONS ?

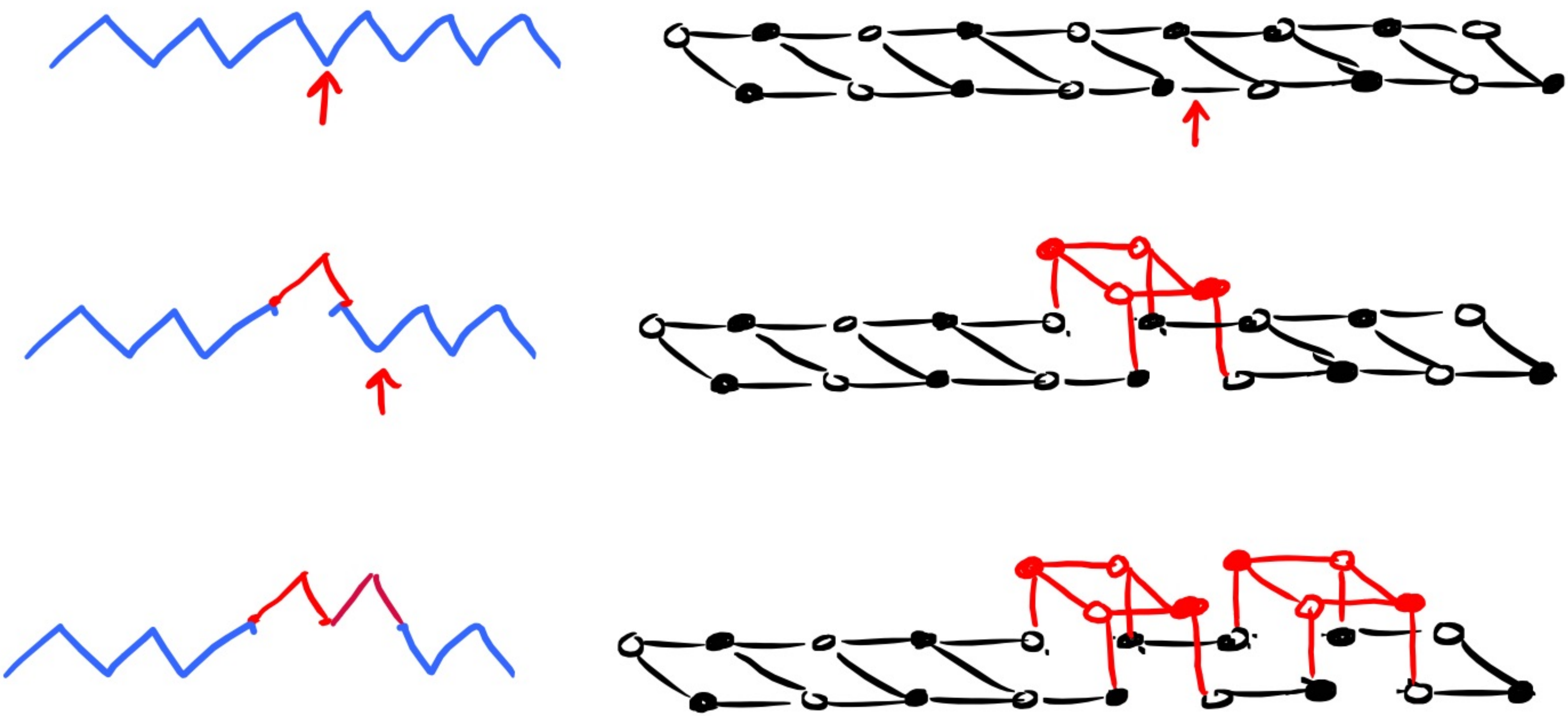
representation in 3D



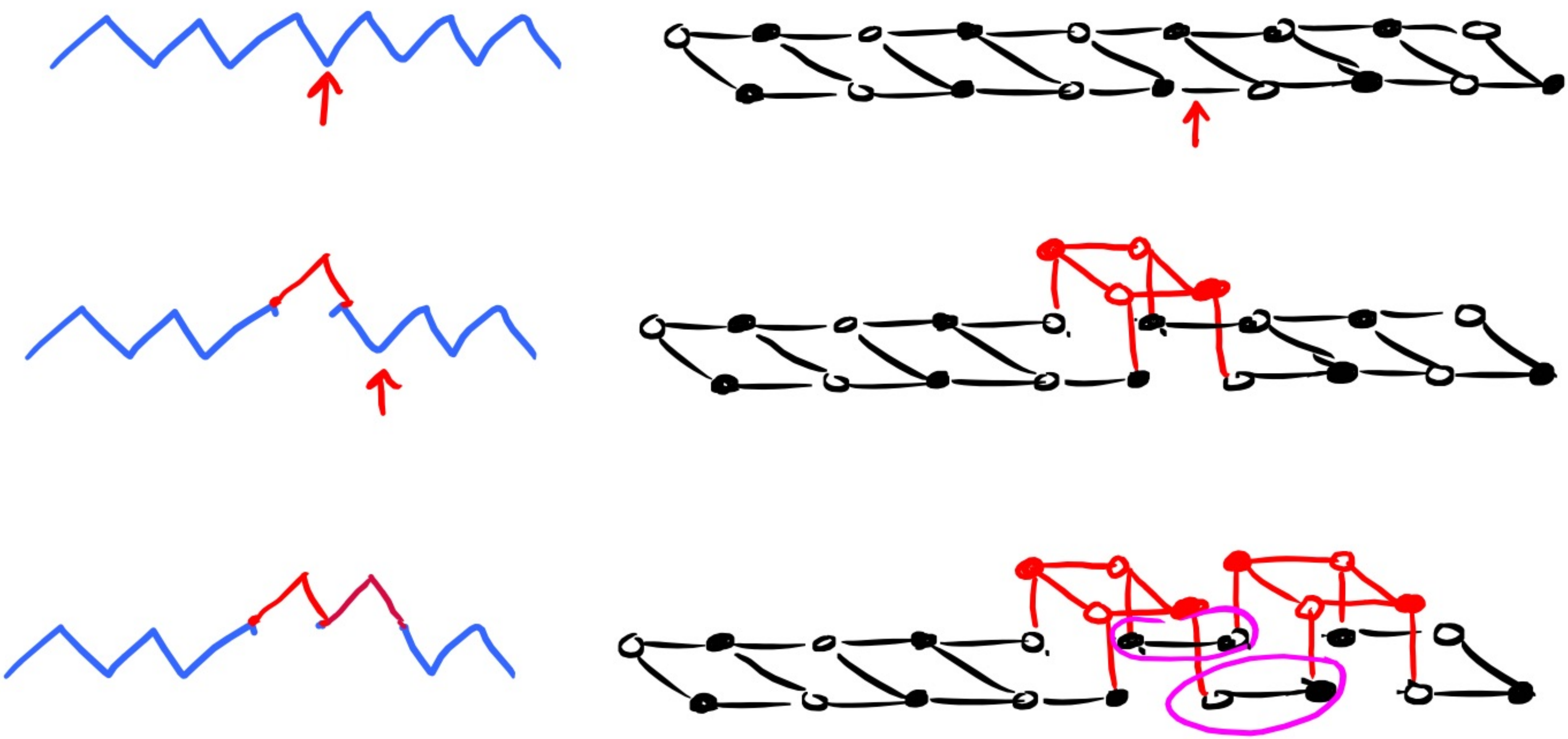
MUTATIONS ? → Urban renewal



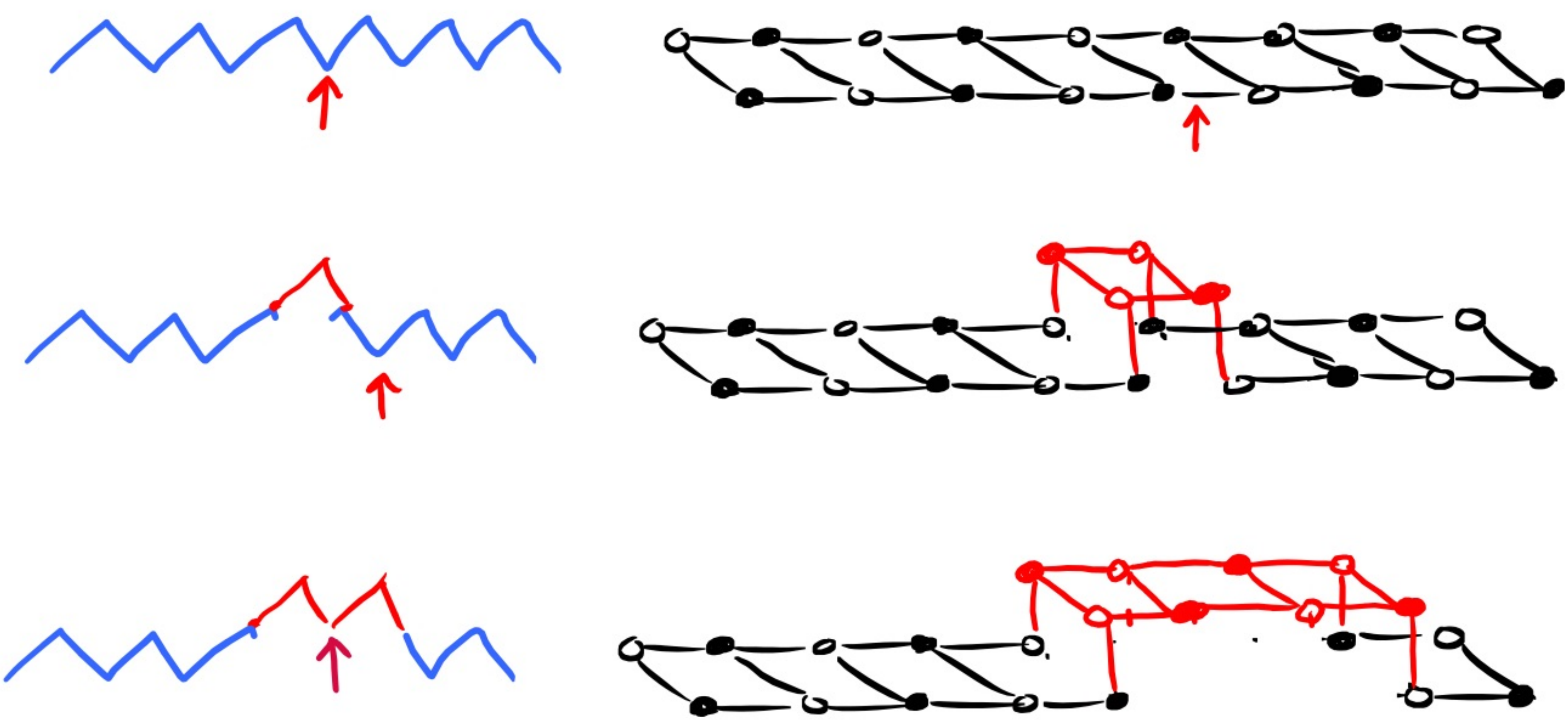
MUTATIONS ?



MUTATIONS ?

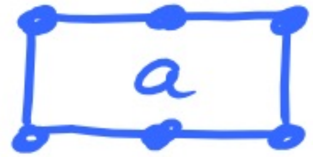


MUTATIONS ?



etc.

weights : $w\Big($  $\Big) = a^{1-d}$

$w\Big($  $\Big) = a^{2-d}$

$d = \#$ dimers around the square / hexagon

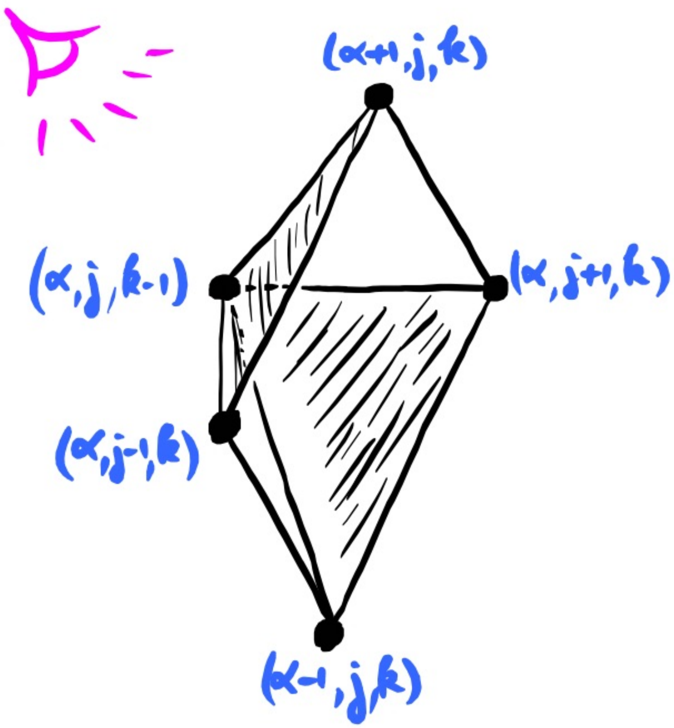
THM for any given initial data

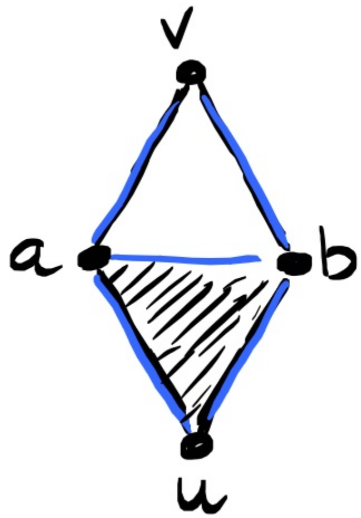
$$T_{ij} = \sum_{\substack{\text{dimers on}\\ \text{3D ladder}\\ \text{graph}}} \prod (\text{face weights})$$

Rem:
"reverse quantum gravity" $Z = \text{invariant}(\text{surface} + \text{weights})$

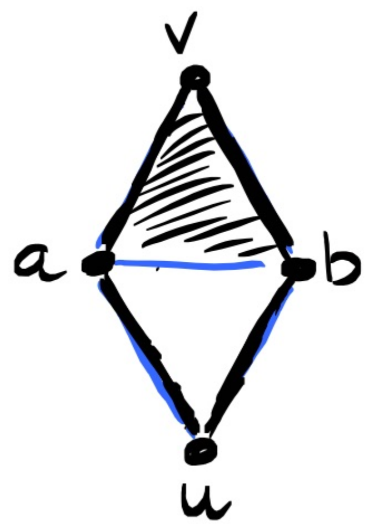
T- system

• same idea = produce a flat connection



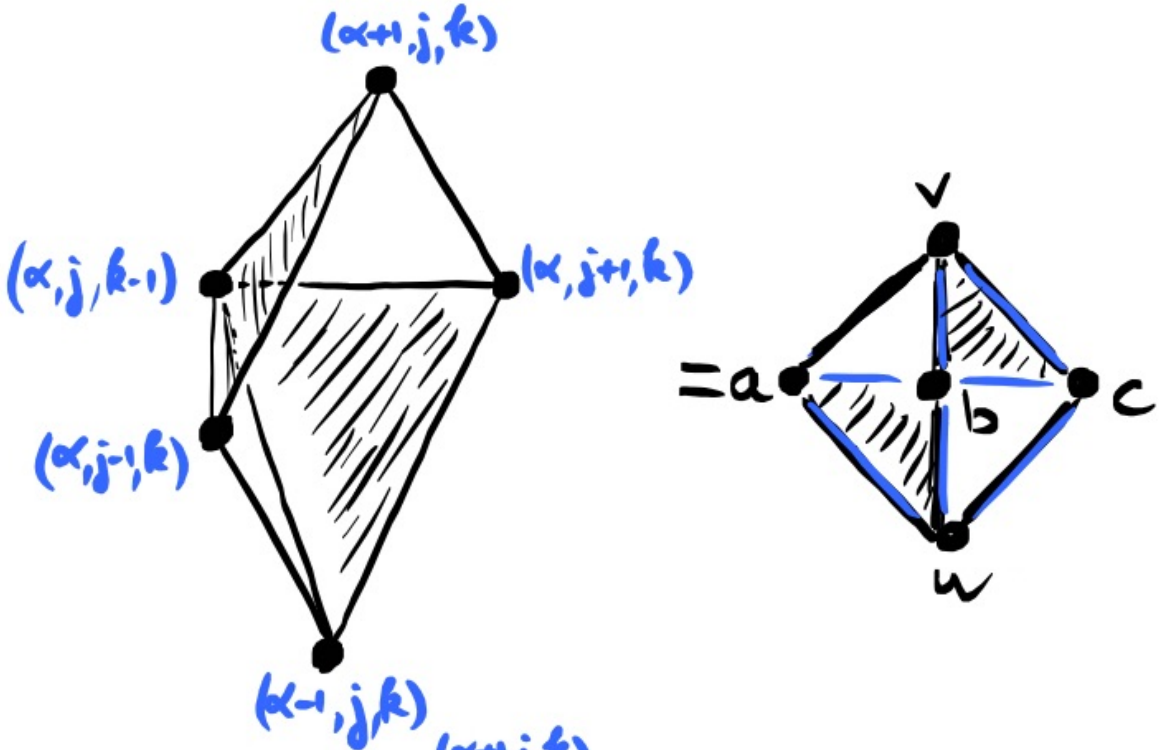


$$\longrightarrow \quad D(u,a,b) = \begin{pmatrix} \frac{a}{b} & \frac{u}{b} \\ 0 & 1 \end{pmatrix}$$

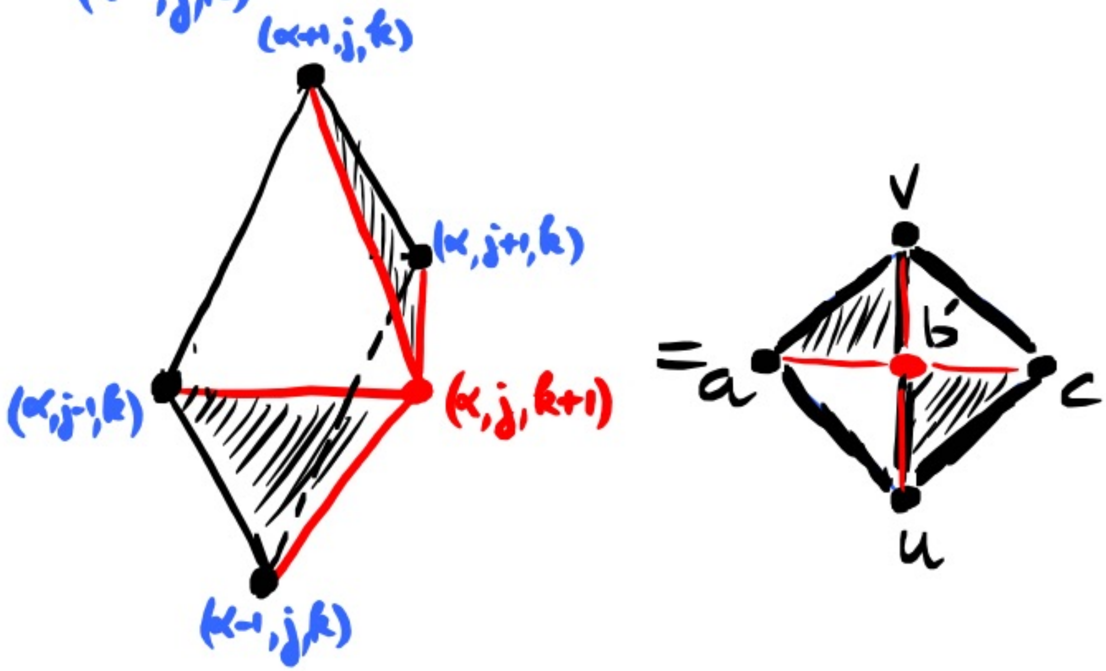


$$\longrightarrow \quad U(a,b,v) = \begin{pmatrix} 1 & 0 \\ \frac{v}{b} & \frac{a}{b} \end{pmatrix}$$

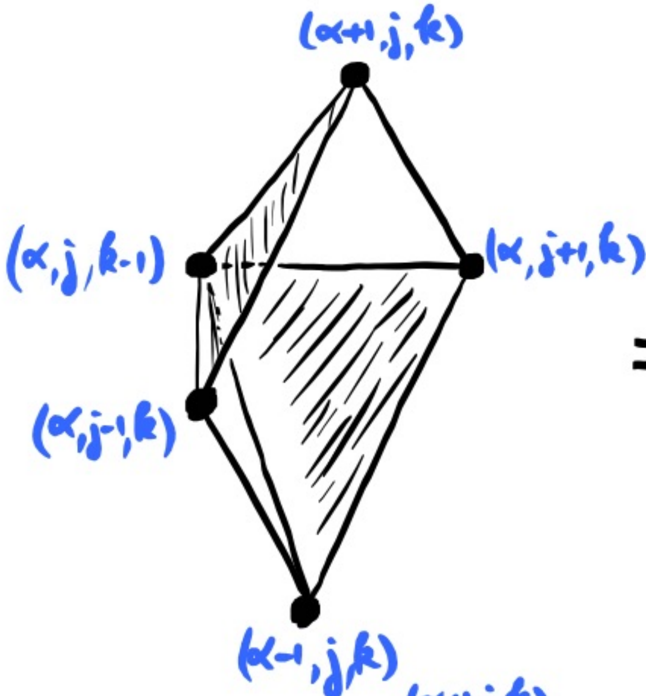
(A_1 case: $u = v = 1$; $\alpha = 1$) .

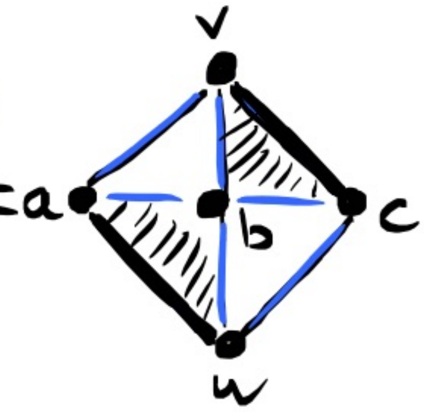


$$= D(u,a,b)\, U(b,c,v)$$



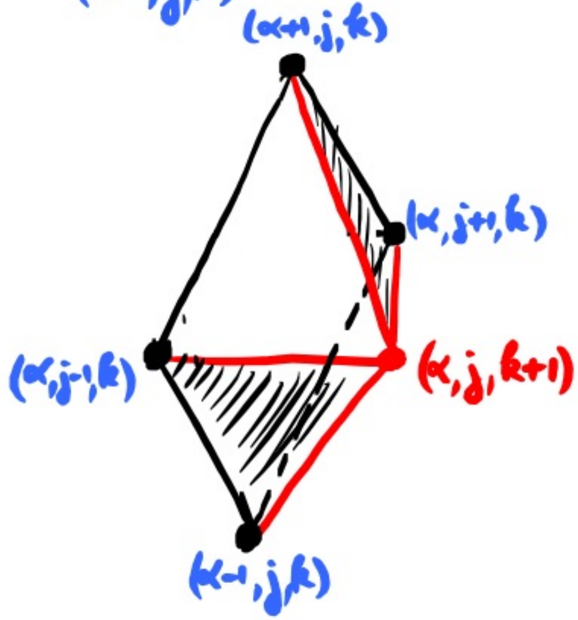
$$= U(a,b',v)\, D(u,b',c)$$

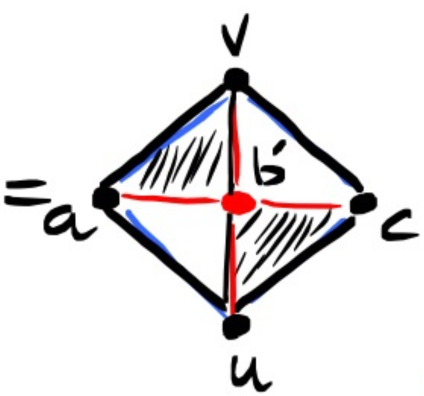


$=$ 

$$= D(u,a,b)\,U(b,c,v)$$

$$\Leftrightarrow \boxed{bb' = ac + uv}$$

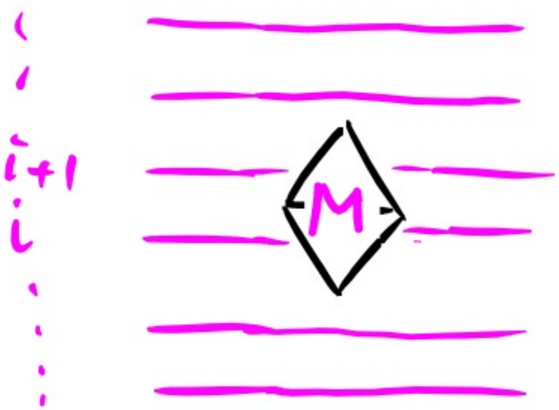


$=$ 

$$= U(a,b',v)\,D(u,b',c)$$

"Flat connection"
(related to Yang-Baxter eq)

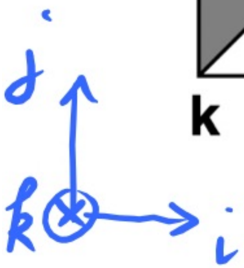
- Attach to the initial data stepped surface a product of D, U matrices:

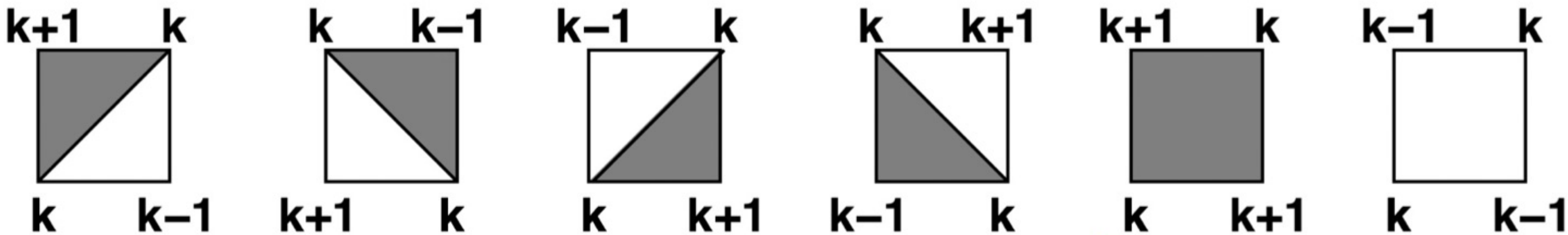
$M_i =$ 

$$= \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} & i & i+1 & \end{array} \\ \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & \boxed{M} & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} \end{array}$$

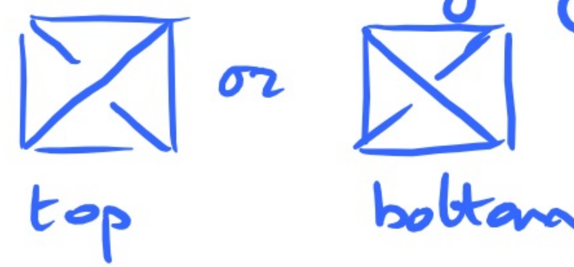
- Product rule: $M_i \cdot P_j$ iff M to the left of P
- well-defined for <u>any initial data stepped surface</u>

Rules:



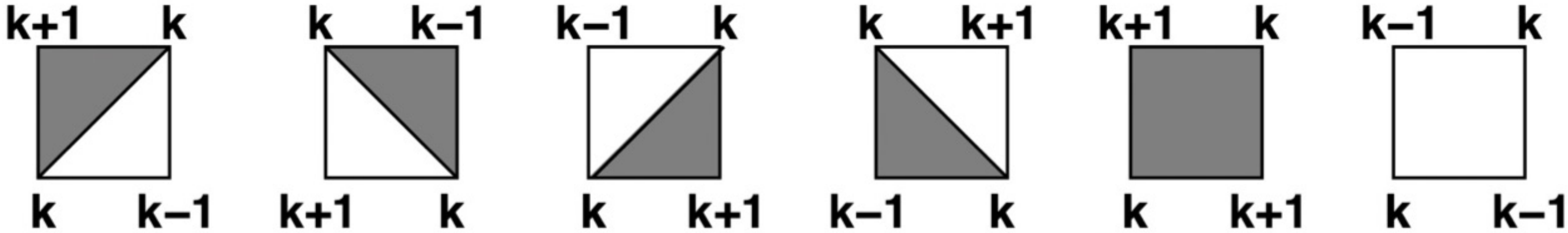


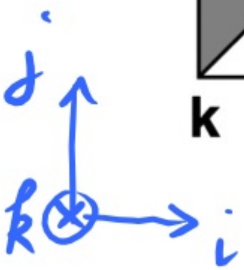
Tetrahedron ambiguity



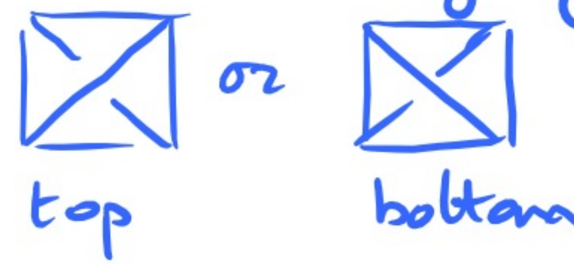
Stepped surface = {vertices}
but Triangulation not unique !

Rules:





Tetrahedron ambiguity



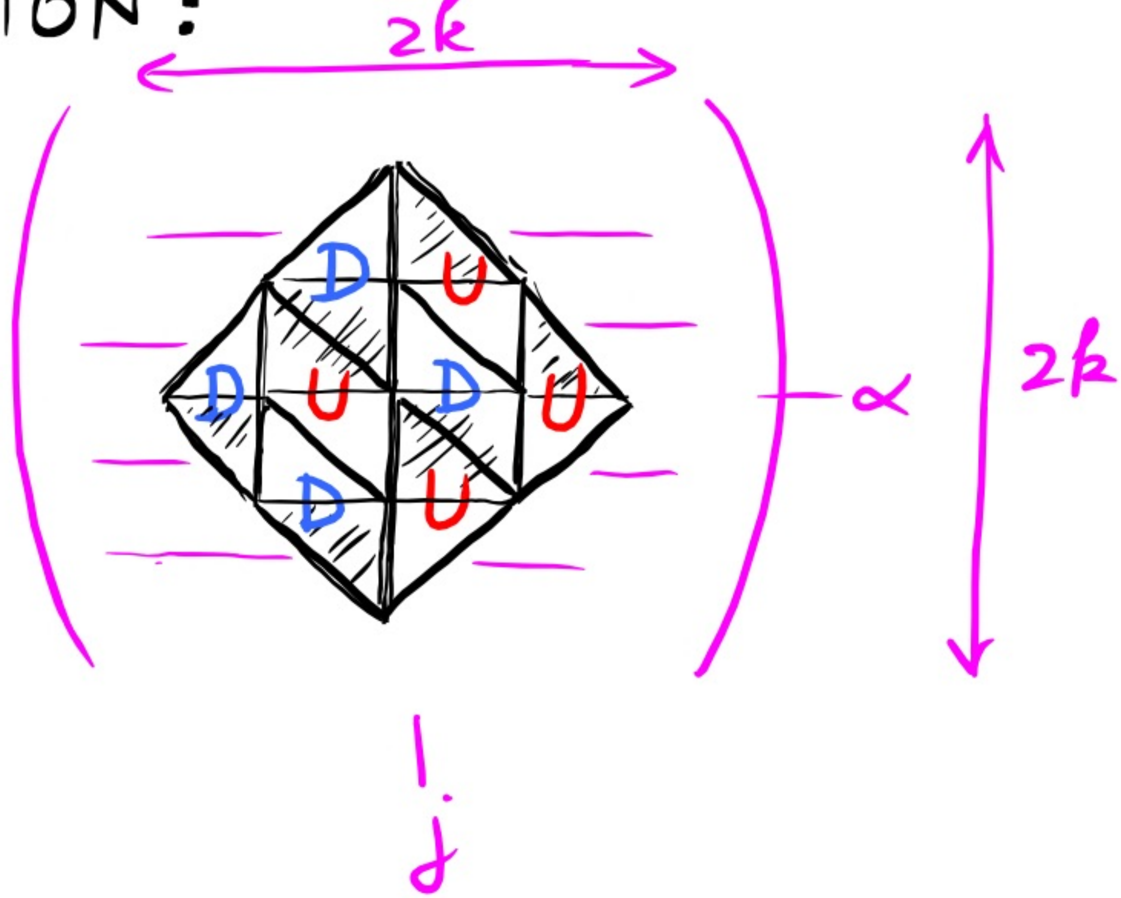
The matrix reps does not see this

$$U_{23} V_{12} = V'_{12} U'_{23} \qquad V_{23} U_{12} = U'_{12} V'_{23}$$

$\Rightarrow$ Matrix product depends only on surface not Triangulation

SOLUTION:

Principal Minor $[1,k]\times[1,k]$

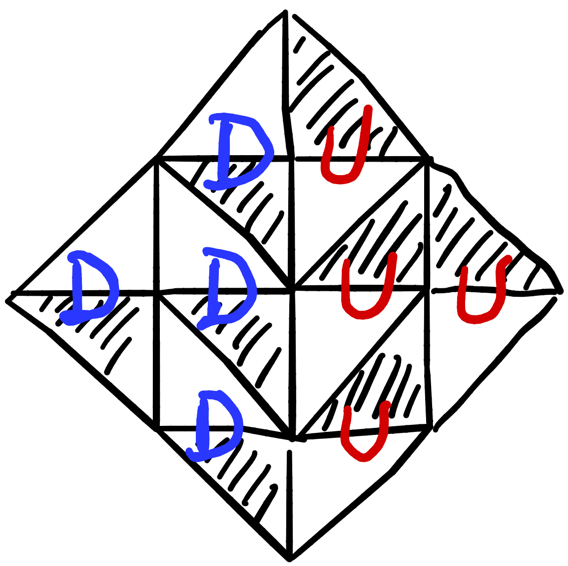


$\sim T_{\alpha,j,k+1}$

POSITIVITY:

entries of D, U are ≥ 0 monomials of initial data $\Rightarrow$ Laurent Positivity

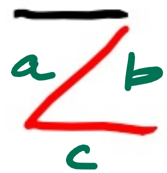
- Proof = the same : compute the quantity in the "pyramid" configuration

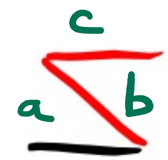


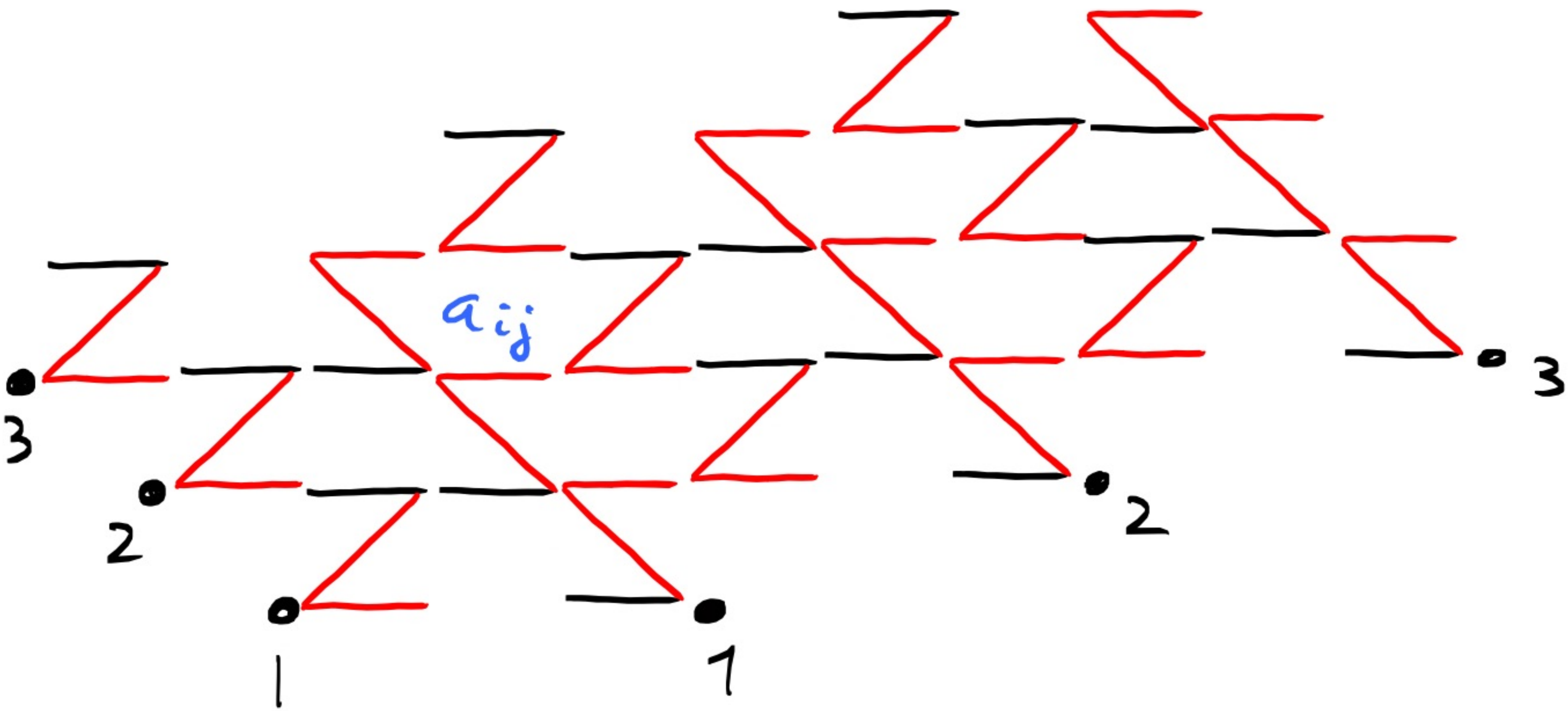
lower triang   upper triang

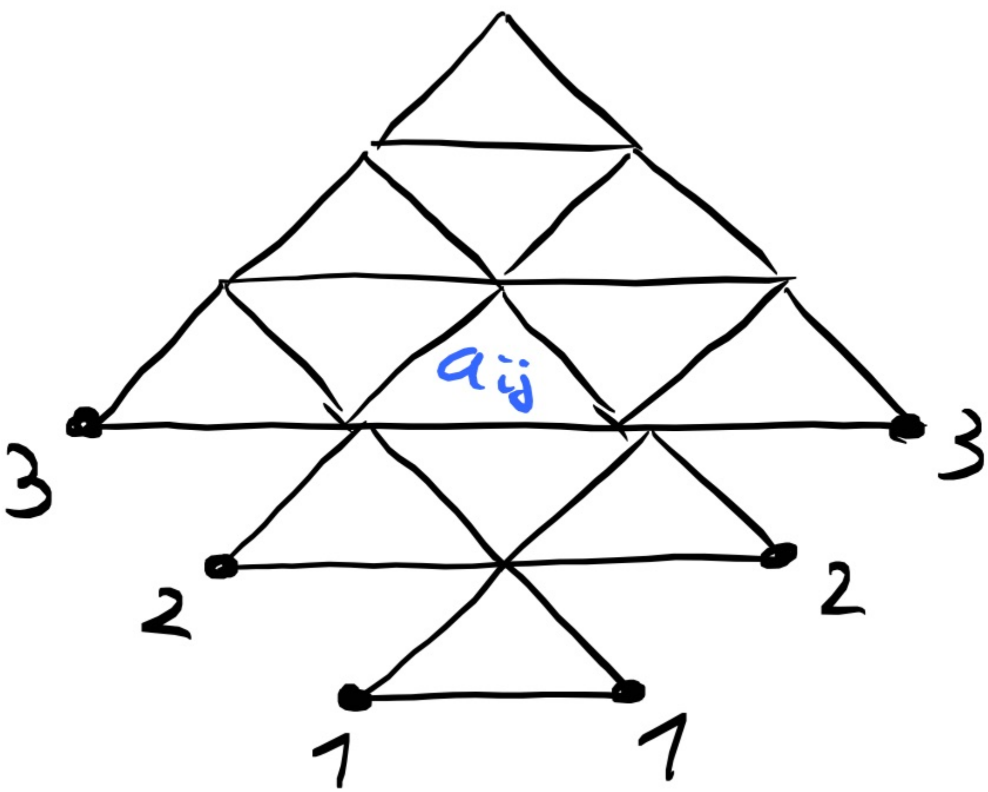
$\Longrightarrow$ analogous telescopic cancellations

NETWORK FORMULATION : flat initial data

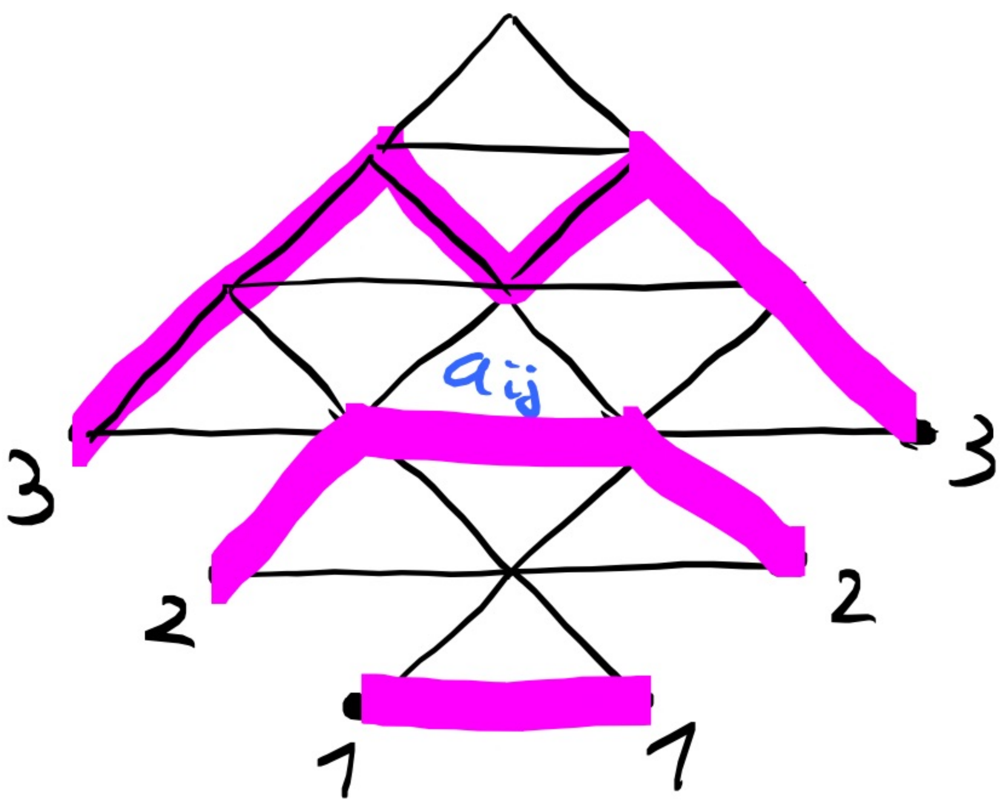
$D_{abc} =$ 

$U_{abc} =$ 





By Gessel Viennot:
principal minor =
$\sum$ non-intersecting
paths $(1,2,..k) \to (1,2,..k)$



By Gessel Viennot:
principal minor =
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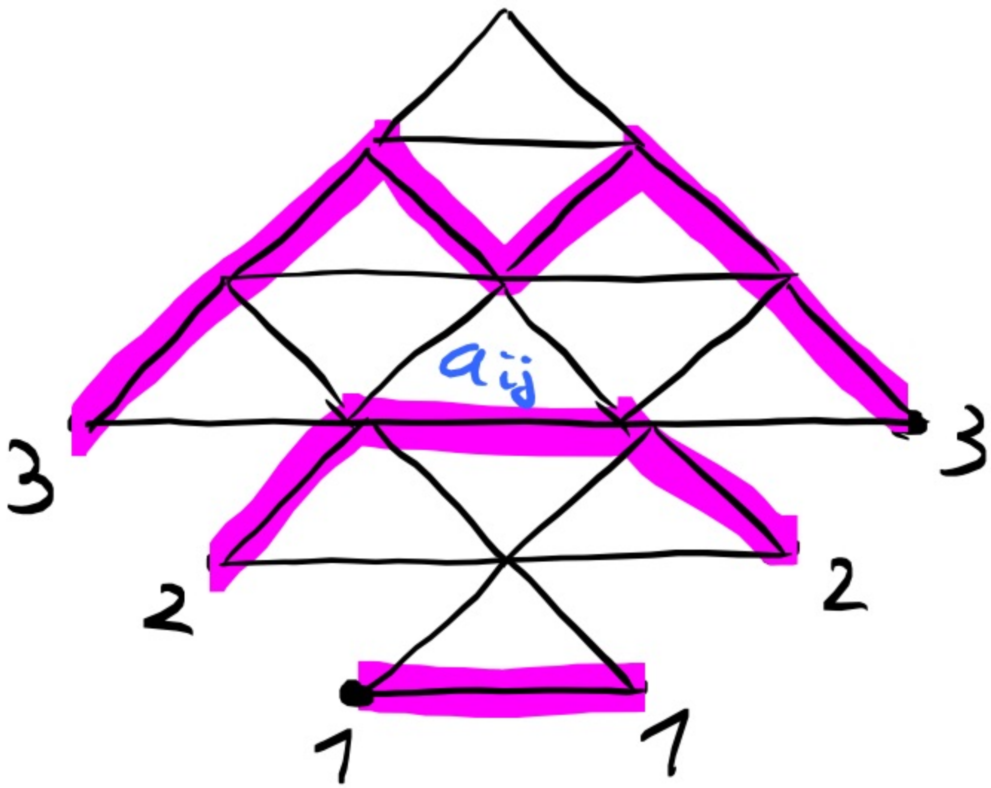
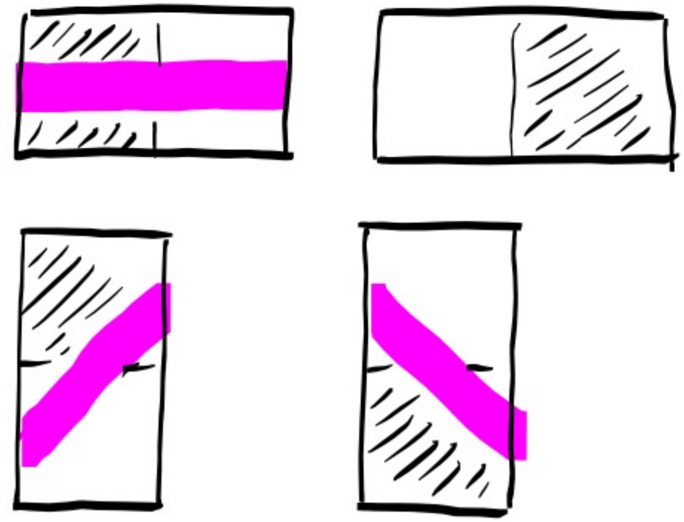
paths $(1,2,..k) \to (1,2,..k)$

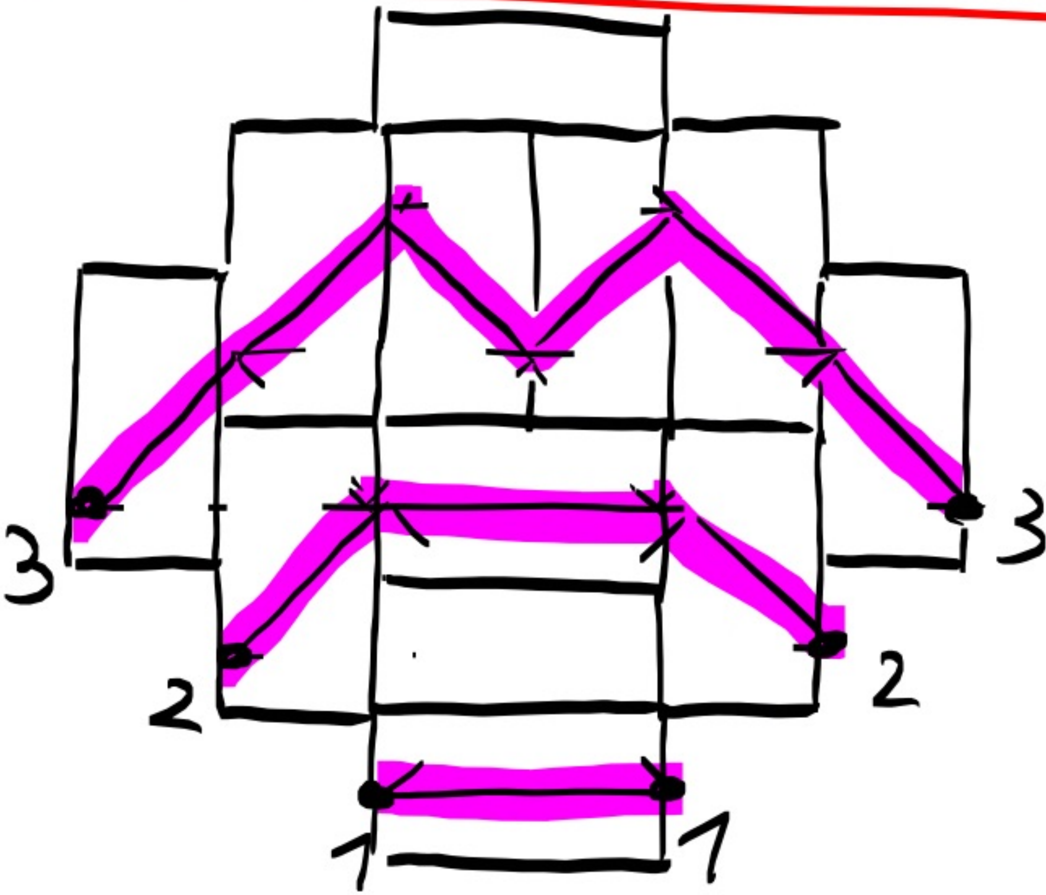
FROM NETWORK PATHS TO DOMINOS

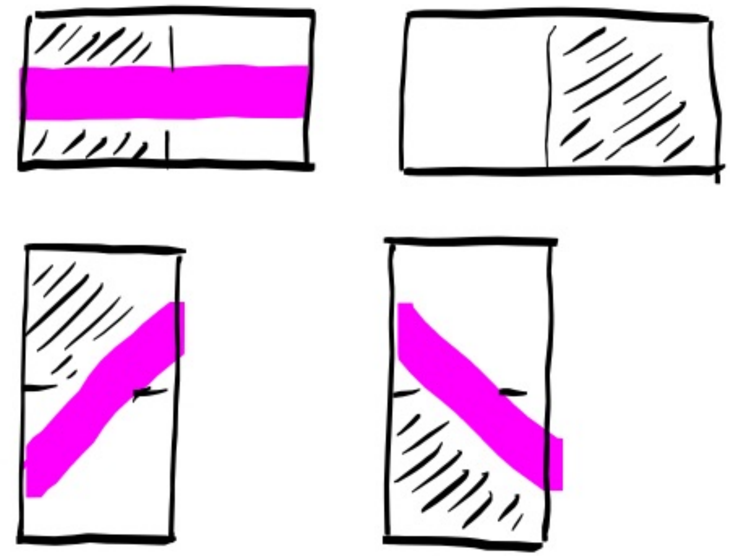




bijection
path – tiling

FROM NETWORK PATHS TO DOMINOS





Domino Tilings of the k×k Aztec Diamond!

(+ weights)

Arctic Circle

- Consider the solution with initial data $\boxed{\begin{array}{r} T_{ij0} = T_{ij1} = 1 \\ T_{001} = x \end{array}}$
- Define $\rho_{ijk} = \frac{\partial}{\partial x} \operatorname{Log} T_{ijk} \Big|_{x=1} = \langle 1 - D_{\infty} \rangle$ (susceptibility)
- Differentiate octahedron eqn : $\frac{\partial}{\partial x}\left(TT = TT + TT \right)\Big|_{x=1}$

Then : $\boxed{2(\rho_{ijk+1} + \rho_{ijk-1}) = \rho_{i+1jk} + \rho_{i-1jk} + \rho_{ij+1k} + \rho_{ij-1k}}$

- Define gen. function $\rho(x,y,z) = \sum_{ijk\geq 0} x^i y^j z^k \rho_{ijk}$

$$\boxed{\rho(x,y,z) = \frac{z}{1 + z^2 - \frac{1}{2} z\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}}$$

- Singularities from the denominator: $\begin{cases} x \to 1 - t x \\ y \to 1 - t y \\ z \to 1 + t(ux + vy) \end{cases}$ $t \to 0$

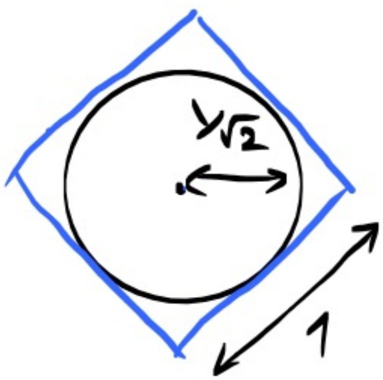
probes $\rho_{uk,vk,k}$ as $k \to \infty$

- Series expansion in t:

$$1 + z^2 - \frac{z}{2}(x + x^{-1} + y + y^{-1}) \approx \frac{t^2}{2}\underbrace{\left(4uvxy + (2u^2-1)x^2 + (2v^2-1)y^2\right)}_{P(x,y)}$$

- Singularity locus: $P(x,y) = 0$ & $\frac{\partial P}{\partial x}(x,y) = 0$

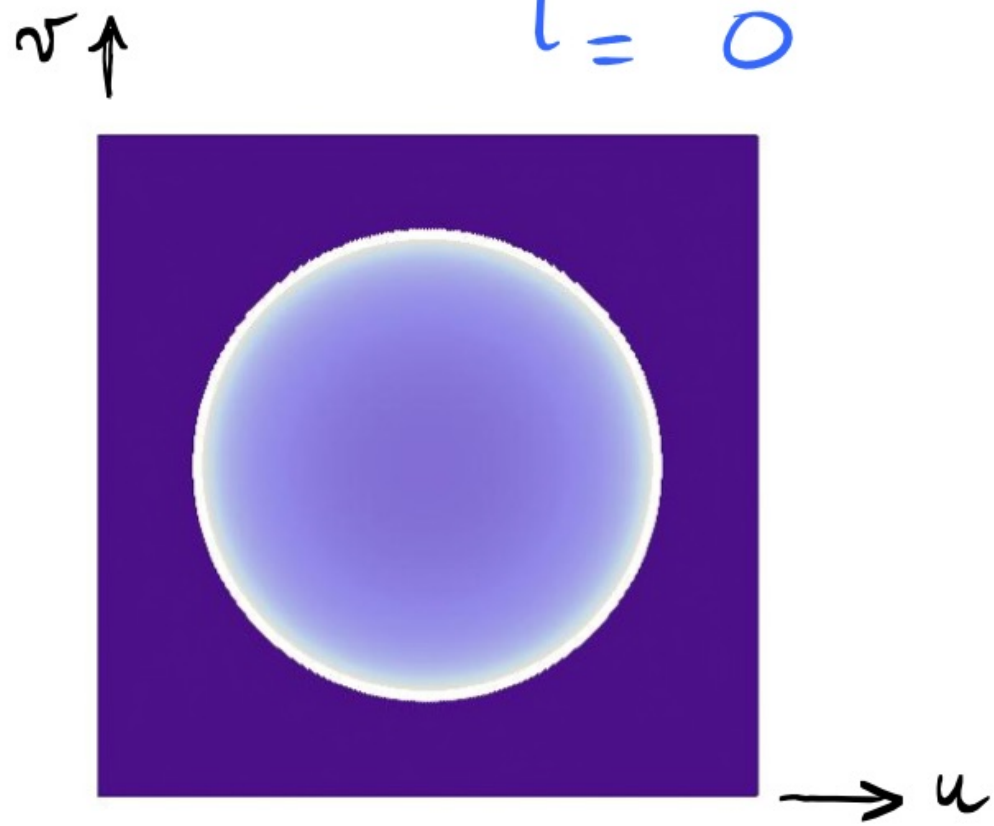
$$\Leftrightarrow \quad \boxed{2(u^2+v^2) - 1 = 0} \quad \text{ARCTIC CIRCLE}$$



other initial data ?

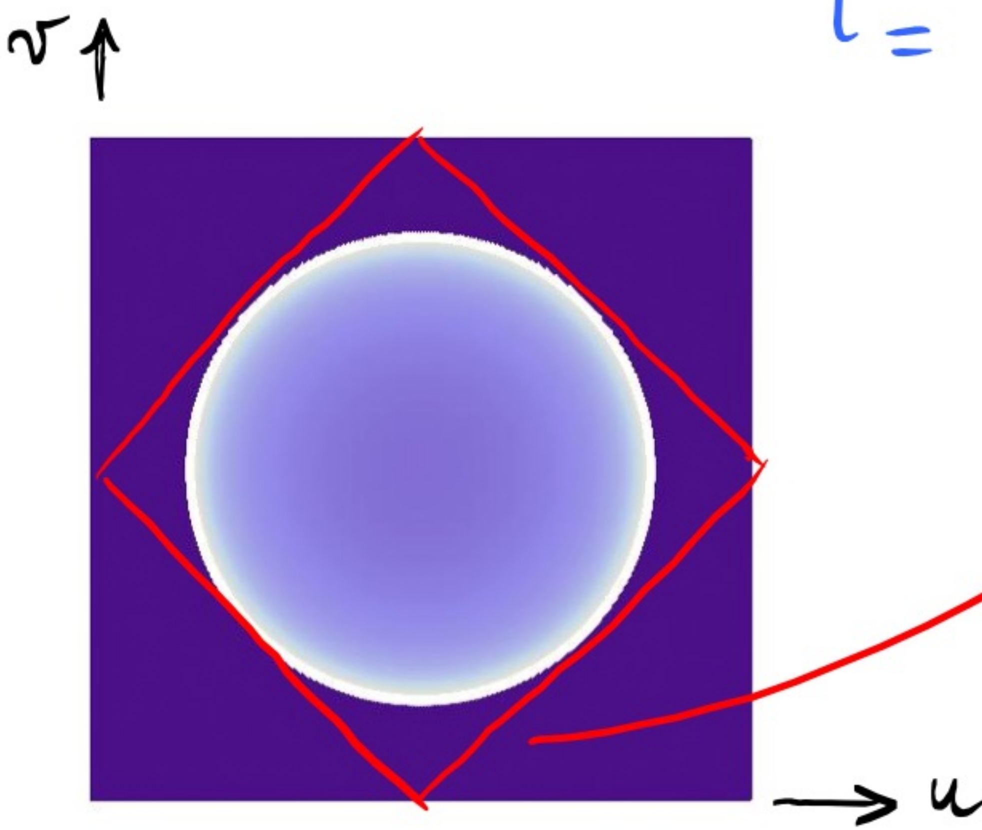
Behavior of ρ_{ijk} for $\frac{i}{k} \sim u$ $\frac{j}{k} \sim v$ $k \to \infty$

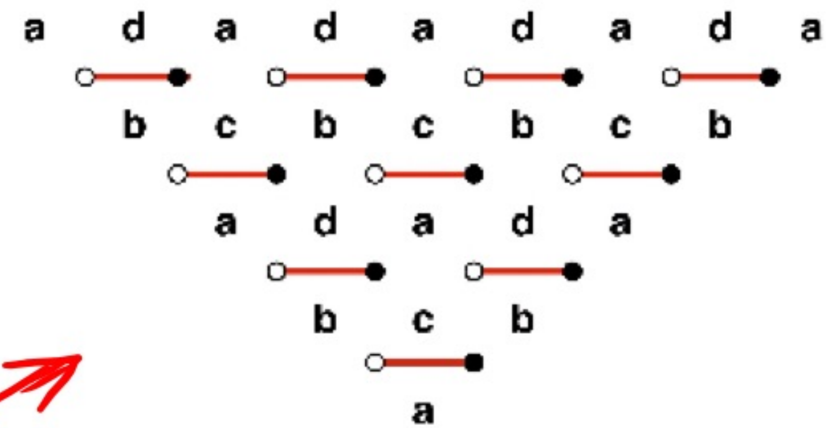
$$\rho(u,v) = \lim_{k\to\infty} k\, \rho_{i,j,k} \begin{cases} = \dfrac{2}{\pi} \dfrac{1}{\sqrt{1-2(u^2+v^2)}} & \left(u^2+v^2 < \frac{1}{2}\right) \\ = 0 & (\text{otherwise}) \end{cases}$$



Behavior of ρ_{ijk} for $\frac{i}{k} \sim u$ $\frac{j}{k} \sim v$ $k \to \infty$

$$\rho(u,v) = \lim_{k\to\infty} k\, \rho_{i,j,k} \begin{cases} = \dfrac{2}{\pi} \dfrac{1}{\sqrt{1-2(u^2+v^2)}} & \left(u^2+v^2<\frac{1}{2}\right) \\ = 0 & (\text{otherwise}) \end{cases}$$

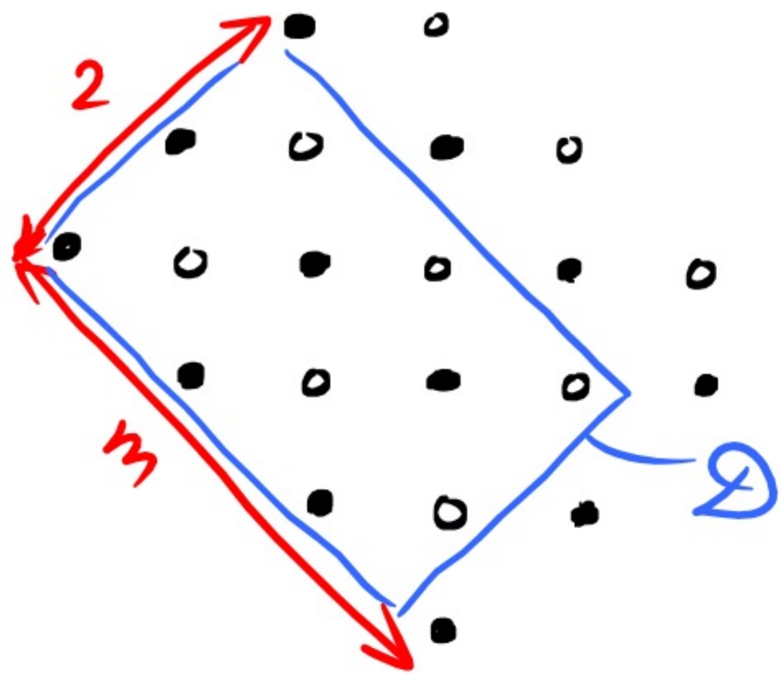




frozen phase (corners)

$\rho = \langle 1-D \rangle = 0$

Periodic initial data: exact solution



$$\begin{cases} T_{i+2,j+2,k} = T_{ijk} \\ T_{i+m,j-m,k} = T_{ijk} \end{cases} \qquad k=0,1$$

- then T_{ijk} = explicit monomial of $\begin{Bmatrix} T_{ij0} & T_{ij1} \\ T_{ij2} & T_{ij3} \end{Bmatrix}$ within the fundamental domain $\mathcal{D}$
- Introduce $\rho_{ijk} = \frac{\partial}{\partial x} \log(T_{ijk}) \Big|_{x=1} \quad (T_{001} = x)$

- Differentiate octahedron eqn $\frac{\partial}{\partial x}\left(TT = TT + TT\right)\Big|_{x=1}$

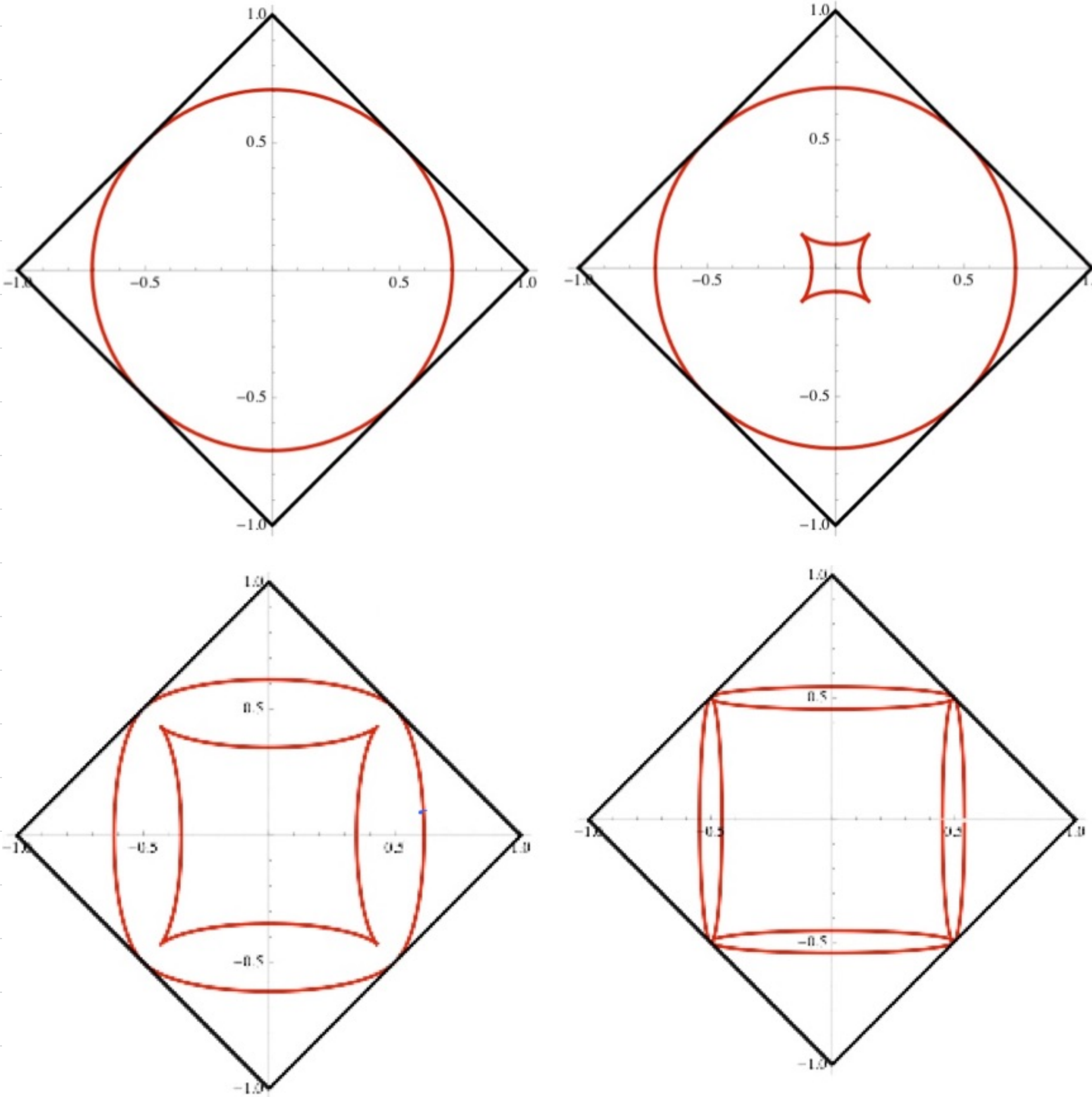
$$\Rightarrow \quad \rho + \rho = \underbrace{\frac{TT}{TT}}_{\mu}(\rho+\rho) + \underbrace{\frac{TT}{TT}}_{1-\mu}(\rho+\rho)$$

$\Rightarrow$ linear recursion for ρ_{ijk} w/ periodic coefficients

$$\boxed{\mu_{ijk} = \mu_{i+2,j+2,k} = \mu_{i+m,j-m,k} = \mu_{i+1,j+1,k+2}}$$

$\Rightarrow$ generating series $\rho(xyz)$ has for denominator the det of a $4m \times 4m$ matrix $\in \mathbb{Z}[x,x^{-1},y,y^{-1},z,z^{-1}]$

Arctic curve = singularity locus.

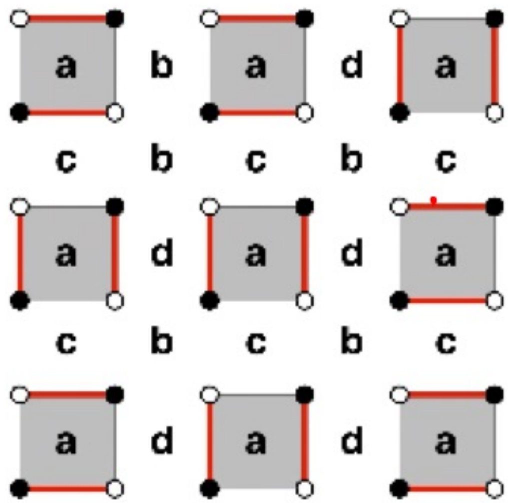


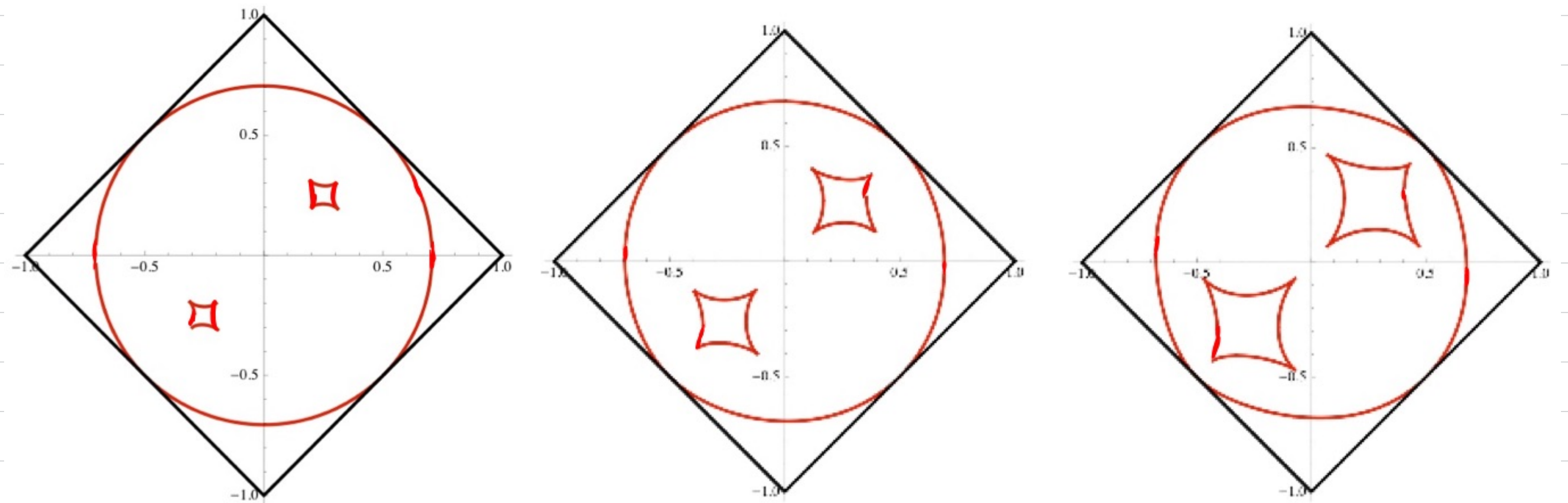
2×2 period case

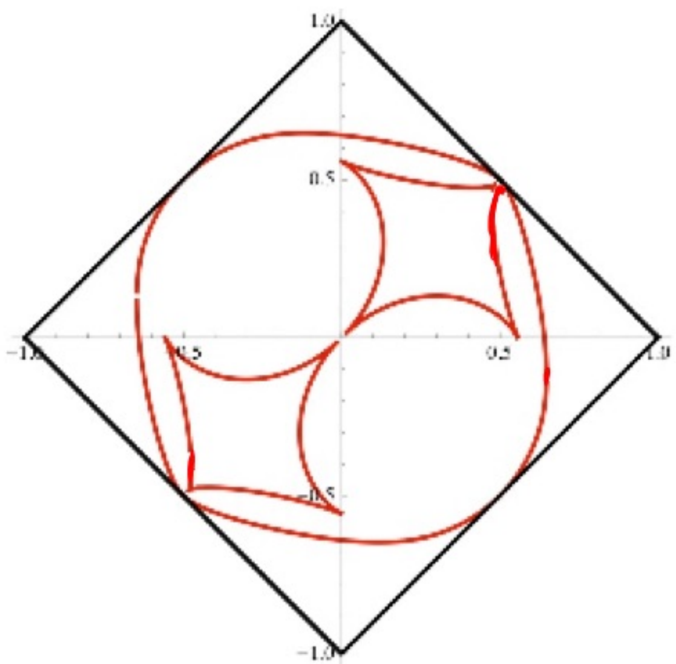
1 parameter

center phase = gas
corner = crystal
intermediate = liquid

value of $g(u,v) = \lim_{k\to\infty} k\, g_{ijk}$: 3 phases $\begin{cases} \text{frozen (corners)} \\ \text{disordered} \\ \text{facet} \end{cases}$



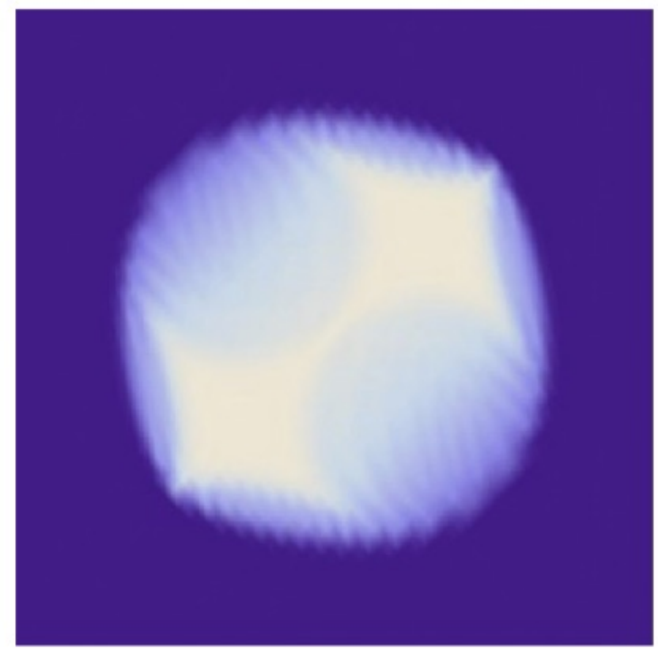




Period 2×3

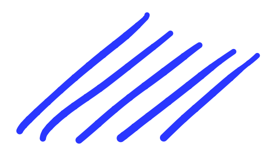
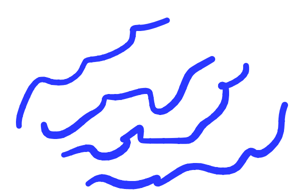
$\left\{\begin{array}{l}\text{2 gaseous phases ("bubbles")} \\ \text{4 crystal phases} \\ \text{1 liquid phase}\end{array}\right.$

value of $\rho(u,v) = \lim_{k \to \infty} k \rho_{ijk}$



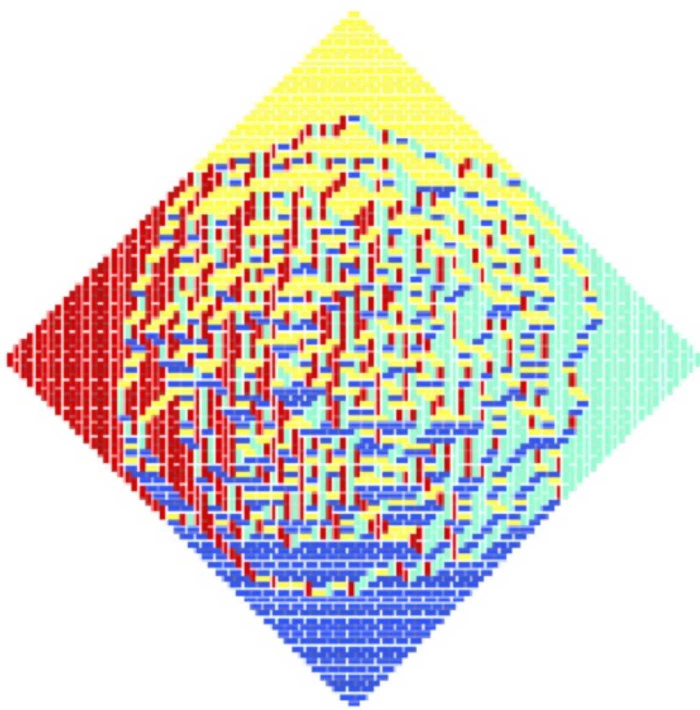
Arctic Curves: a path-based approach

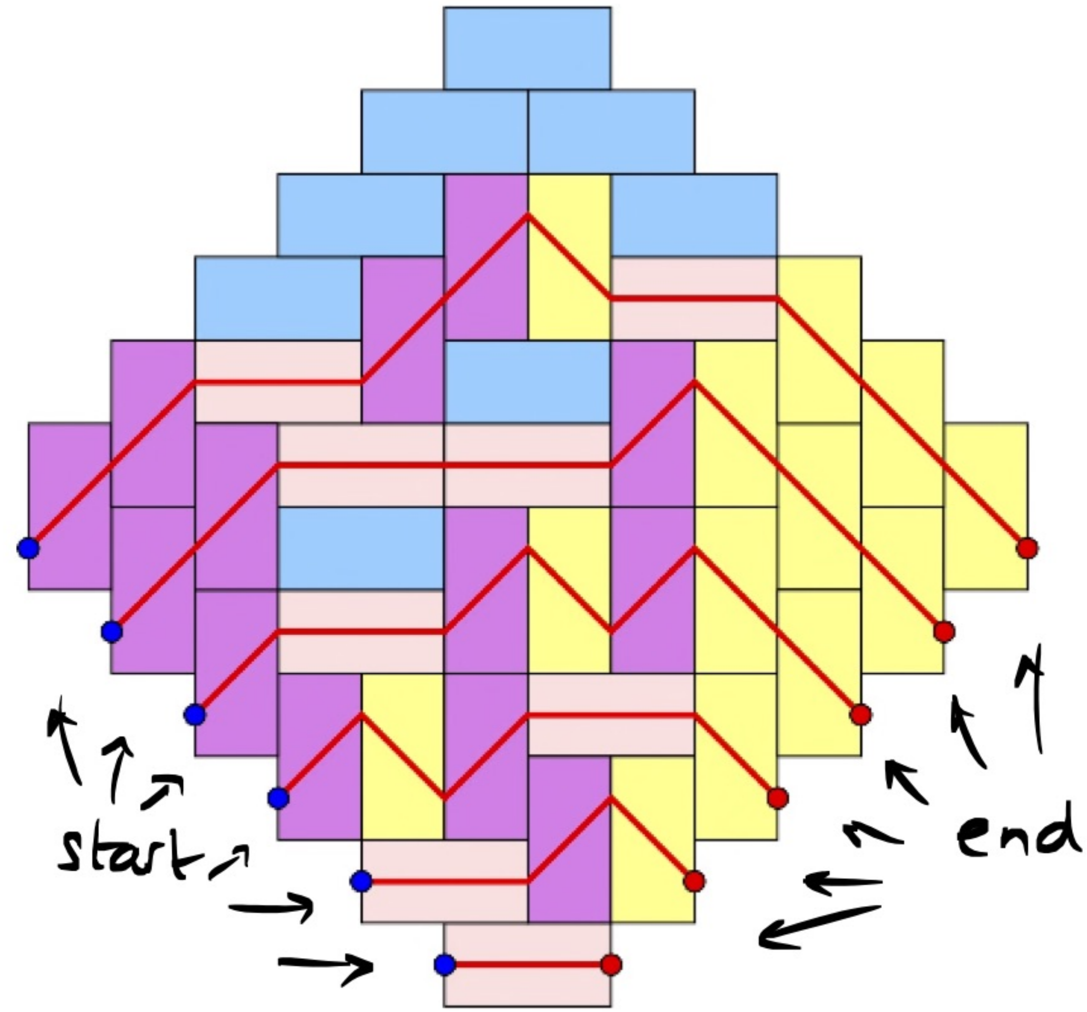
- Start from a formulation in terms of non-intersecting lattice paths

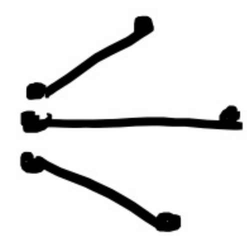
PHASES
- frozen = regular  or no path
- disordered 

- arctic curve = sharp separation between phases

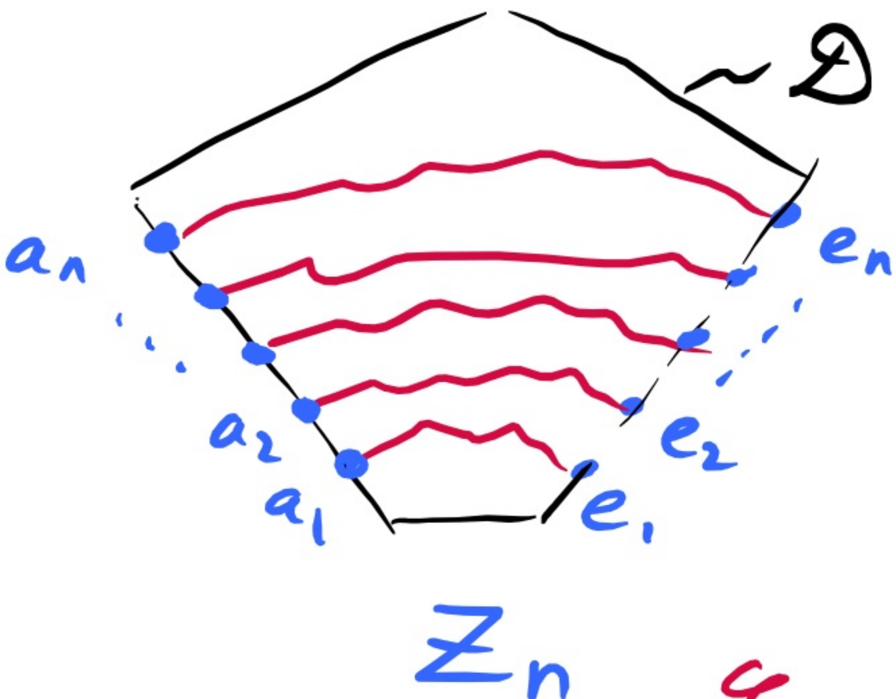
Our case study: domino tilings of the Aztec Diamond



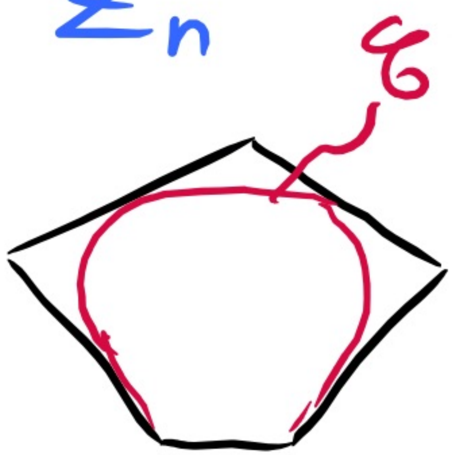


Schröder Paths: $\{$  $\}$

The tangent method

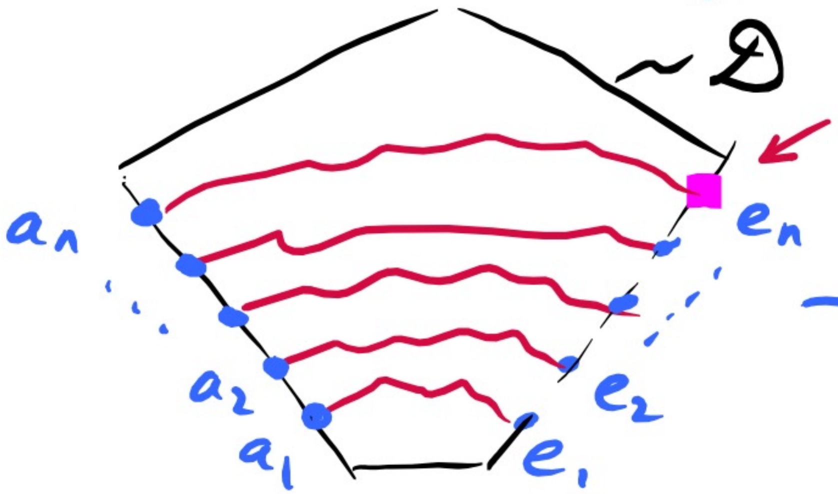


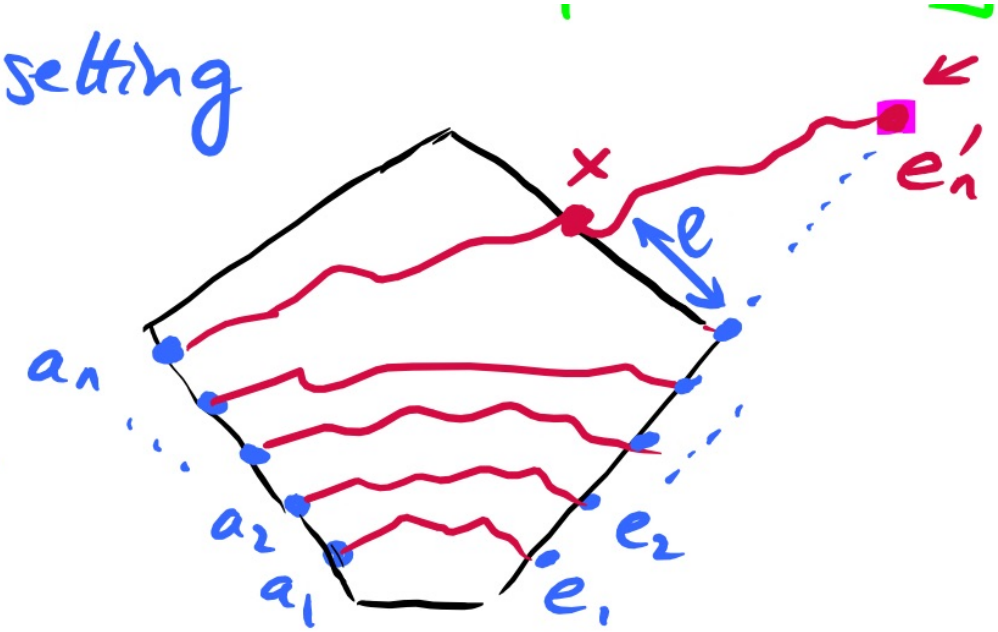
- large size



some portion of $\mathcal{C}$ is the limiting outer envelope of the path sap

- Change the setting

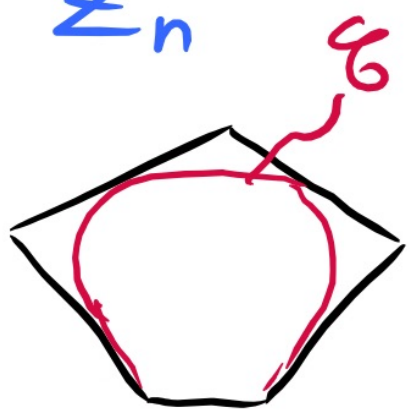


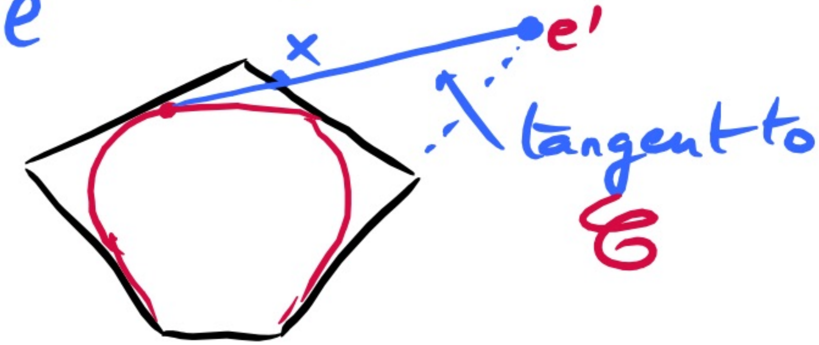


Z_n

$$\sum_e Z_{n,e} \times Z_{x \to e'_n}$$

- large size





- move the outer path end
- use it as probe of $\mathcal{C}$

The tangent method:

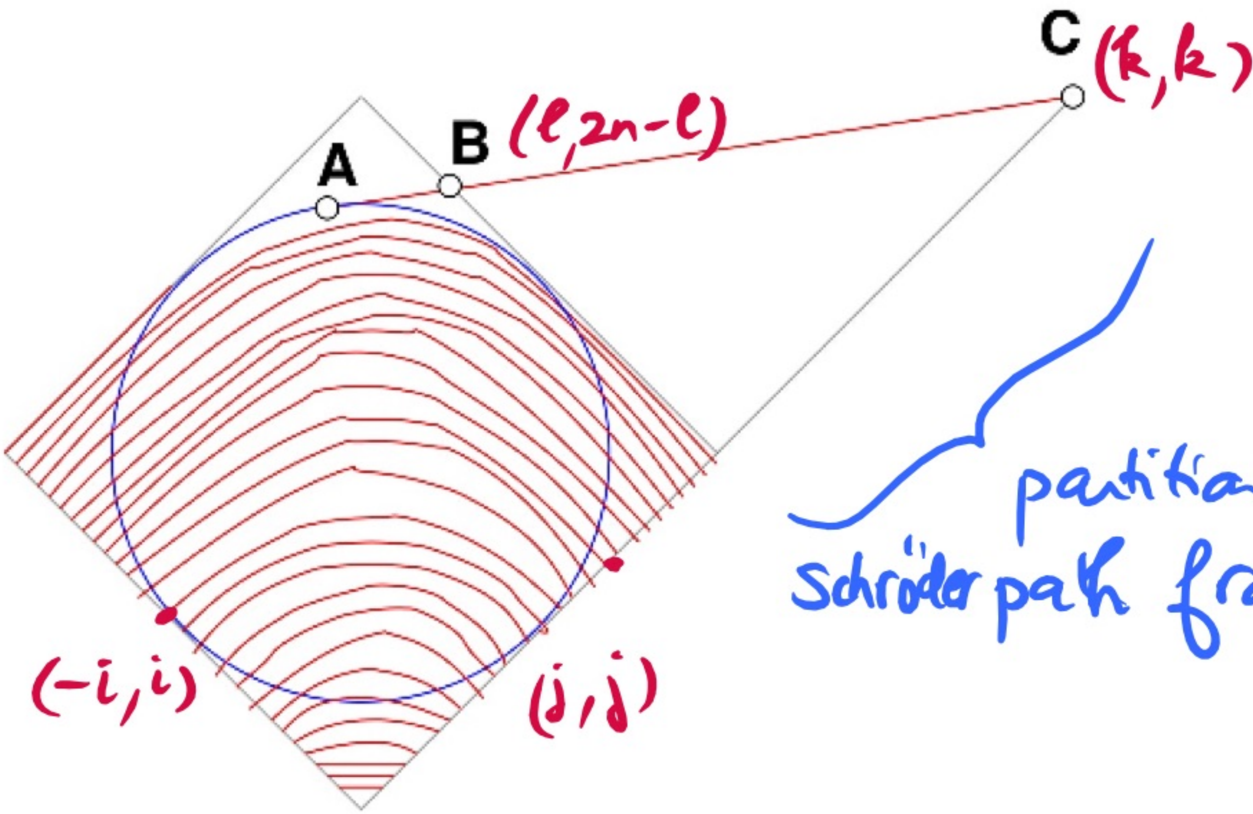
- Relies on 2 properties:

1. "left to its own devices, a directed random path with fixed endpoints is most likely to follow a straight line"

2. "The line followed by the external path away from the others is tangent to the arctic curve $\mathcal{C}$"

1. can be proved rigorously.
2. still an assumption

Application to the domino tilings of the Aztec diamond



partition function for a single Schröder path from B → C $= Y_{\ell,k}$

partition fctn w/escaping path at B

" $Z_{n,\ell}$

$$\underbrace{\frac{1}{Z_n}}_{\text{normalization}} \sum_{\ell} Z_{n,\ell}\, Y_{\ell,k} \quad ?$$

normalization by $Z_n = Z_{n,n}$ (partition function of the DT)

most likely ℓ ?

Z_n)

Computing the partition function

LGV matrix:

$$A_{ij} = \frac{1}{1-z-w-zw}\Bigg|_{z^i w^j} = \sum_{p=0}^{\mathrm{Min}(i,j)} \frac{(i+j-p)!}{p!\,(i-p)!\,(j-p)!}$$

LU decomposition

$A = L \cdot U$ (L = uni-lower ; U = upper).

$$L_{ij} = \frac{1}{1-z(1+w)}\Bigg|_{z^i w^j} = \binom{i}{j} \qquad (L^{-1})_{ij} = (-1)^{i+j}\binom{i}{j}$$

$$U_{ij} = \frac{1}{1-w(1+2z)}\Bigg|_{z^i w^j} = 2^i \binom{j}{i}$$

Partition function:

$$\boxed{Z_n = \det A = \prod_{0}^{n} U_{ii} = 2^{n(n+1)/2}}$$

$Z_{n,\ell}$

LGV matrix: $\tilde{A}_{i,j} = \begin{cases} A_{ij} & j<n \\ A_{i+\ell-n,n} & j=n \end{cases}$ almost same as A

LU decomposition: $L^{-1}\tilde{A} = \tilde{U}$ → same L as A !

$$\tilde{U}_{ij} = \begin{cases} U_{ij} & j<n \\ \sum_{l}^{n} L^{-1}_{ik} \tilde{A}_{kn} & j=n \end{cases}$$ → almost same U

1-pt function:

$$\boxed{H_{n,\ell} = \frac{\det \tilde{A}}{\det A} = \frac{\tilde{U}_{n,n}}{U_{n,n}} = \frac{1}{2^n} \sum_{j=0}^{\ell} \binom{n}{j}}$$

Proof:

$$\tilde{U}_{n,\ell} = \sum_i \underbrace{L^{-1}_{n,i}}_{(-1)^{n+i}\binom{n}{i}} \tilde{A}_{i,n} = \sum_i L^{-1}_{n,i} \underbrace{A_{i+\ell-n,n}}_{\frac{1}{1-z-w-zw}\Big|_{z^{i+\ell-n}w^n}}$$

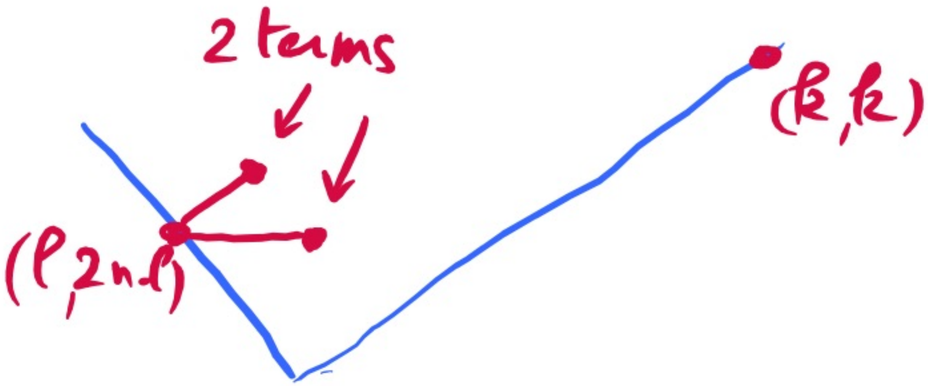
$$= \sum_i (-z)^{n-i}\binom{n}{i} \frac{1}{1-z-w-zw}\Bigg|_{z^\ell w^n}$$

$$= \frac{(1-z)^n}{1-z-w-zw}\Bigg|_{z^\ell w^n} = (1-z)^n \frac{(1+z)^n}{(1-z)^{n+1}}\Bigg|_{z^\ell} = \frac{(1+z)^n}{1-z}\Bigg|_{z^\ell}$$

$$= \sum_0^\ell \binom{n}{j} \qquad \text{qed.}$$

$\Upsilon_{\ell,k}$

Single path from $(\ell, 2n-\ell) \longrightarrow (k,k)$ <u>exiting diamond</u>



$$\boxed{\Upsilon_{\ell,k} = A_{n-\ell,\,k-n-1} + A_{n-\ell-1,\,k-n-1}}$$

Tangent method: asymptotics of $\sum_{\ell} H_{n,\ell} Y_{\ell,k}$

Scaling: $\boxed{n \text{ large} \quad \ell = n\xi \quad k = nz \quad \xi \in (0,1);\ z > 1.}$

$Y_{n,k}$ $$Y_{n\xi,nx} \sim 2A_{n(1-\xi),n(z-1)} \sim \int_0^{\mathrm{Min}(1-\xi,x-1)} d\theta \; e^{S_0(\theta,\xi,z)}$$

$$\boxed{\underset{\text{Stirling}}{S_0(\theta,\xi,z)} = (z-\xi-\theta)\log(z-\xi-\theta) - \theta\log\theta - (1-\xi-\theta)\log(1-\xi-\theta) - (z-1-\theta)\log(z-1-\theta)}$$

$H_{n,\ell}$ $$H_{n,n\xi} \sim \frac{1}{2^n}\sum_{j=0}^{\ell}\binom{n}{j} \sim \int_0^{\xi} d\varphi \; e^{nS_1(\varphi,z)}$$

$$\boxed{S_1(\varphi,z) = -\varphi\log\varphi - (1-\varphi)\log(1-\varphi) - \log 2}$$

Saddle point:

maximize $\int e^{n(S_0+S_1)}$

total action: $S = S_0 + S_1 \quad (\varphi, \theta, \zeta, z)$

$$\frac{\partial S}{\partial \varphi} = 0 \implies \varphi_0 = \frac{1}{2} \qquad (\text{for } 0<\varphi<\zeta)$$

$$\begin{cases} (1)\ \zeta > \frac{1}{2} \quad \text{then} \quad S_1(\varphi_0, z) = 0 \ (\varphi_0 = \frac{1}{2}) \text{ and } H_{n,n\zeta} \sim 1 \\ (2)\ \zeta < \frac{1}{2} \quad \text{then} \quad S_1(\zeta, z) \text{ dominates} = -\zeta \log \zeta - (1-\zeta)\log(1-\zeta) - \log 2 \\ \qquad\qquad (\varphi_0 = \zeta) \end{cases}$$

$$\begin{cases} (1)\ \zeta > \frac{1}{2} \quad S = S_0(\theta, \zeta, z) \\ (2)\ \zeta < \frac{1}{2} \quad S = S_0(\theta, \zeta, z) + S_1(\zeta, z) \end{cases}$$

Now extremize S over θ, ζ : $\frac{\partial S}{\partial \theta} = \frac{\partial S}{\partial \zeta} = 0$

(1) no solution

(2) $(1-\zeta-\theta)(z-1-\theta) = \theta(z-\zeta-\theta)$ and $(1-\zeta-\theta)(1-\zeta) = (z-\zeta-\theta)\zeta$

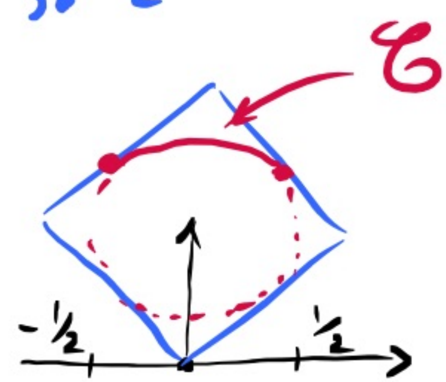
$\Rightarrow$ $\boxed{\zeta_0(z) = \frac{1}{2z}}$ most likely exit point $= (\zeta_0 \,, 2-\zeta_0)$

Tangent Family:

$$L(x,y) = y - \frac{2-\zeta_0-z}{\zeta_0-z}\, x + 2z\, \frac{1-\zeta_0}{\zeta_0-z} = 0$$

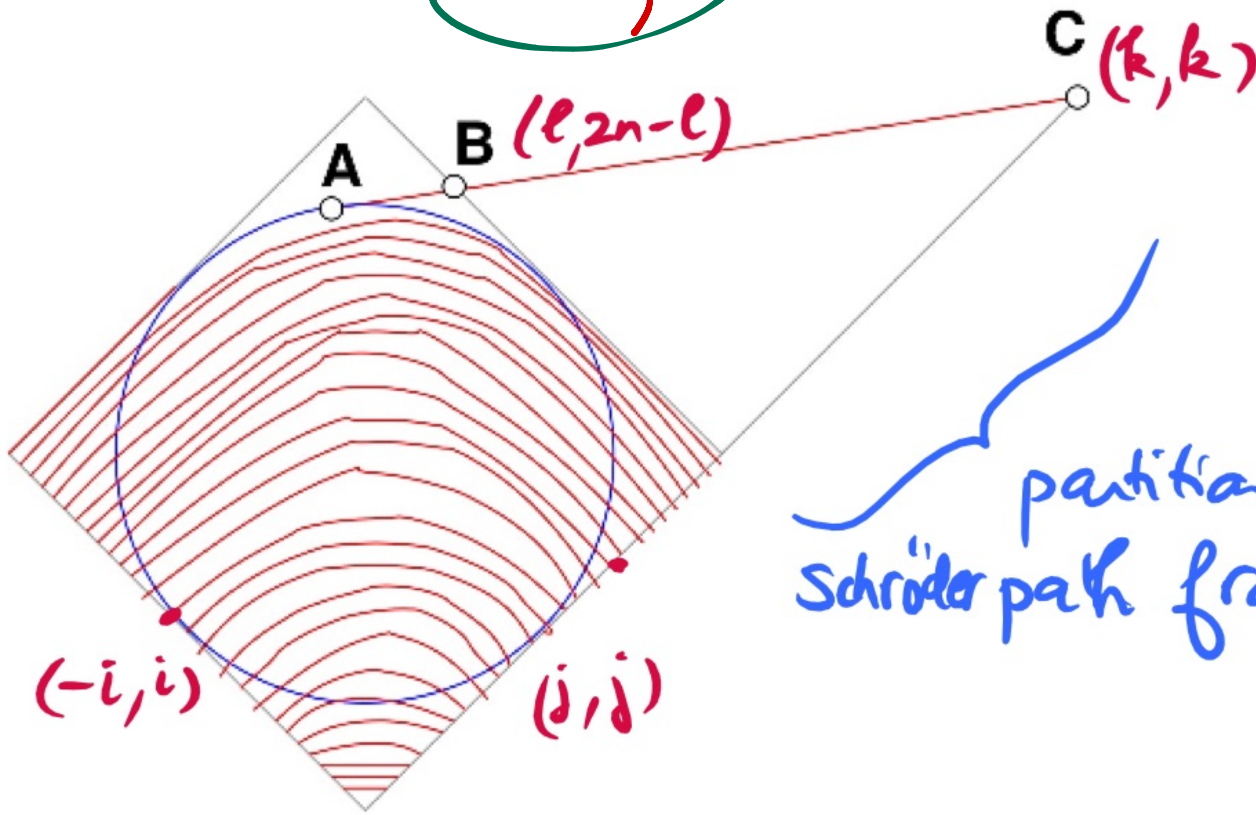
Envelope $\frac{\partial L}{\partial z} = L = 0$

$\mathcal{E}$: $\boxed{x^2 + (y-1)^2 = \frac{1}{2}}$ $x \in (-\frac{1}{2}, \frac{1}{2})$



qed!

$\ell = n\zeta$



$k = nz$

partition function for a single Schröder path from $B \to C$ $= Y_{\ell,k}$

partition fctn w/escaping path at B
"$Z_{n,\ell}$"

$$\frac{1}{Z_n} \sum_{\ell} Z_{n,\ell} \, Y_{\ell,k} \quad ?$$

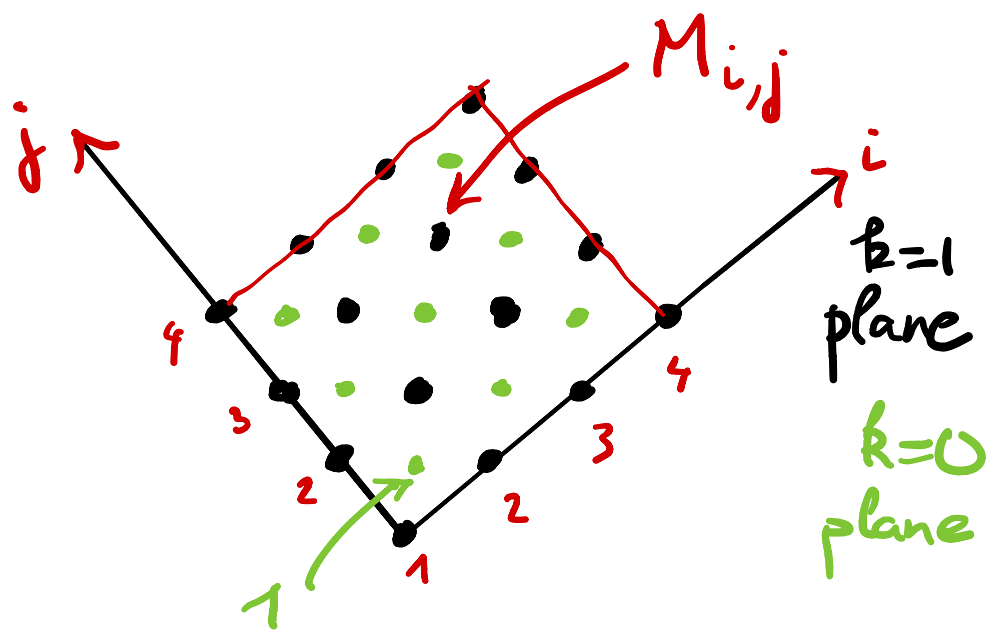
most likely ℓ ?

Conclusion: arctic curves, beyond dimers

- The T-system solution with flat initial data

$$T_{i,j,0} = 1 \qquad (i+j = 0 \bmod 2)$$

$$T_{i,j,1} = M_{\frac{i+j+1}{2}, \frac{i-j+1}{2}} \qquad (i+j = 0 \bmod 2)$$



Thm $$T_{k,0,k} = \sum_{k\times k \text{ ASM}, A} 2^{\#(-1)_A} \prod_{i,j=1}^{k} M_{i,j}^{A_{ij}}$$

where the sum extends over Alternating Sign Matrices A of size $k\times k$ (ASM)

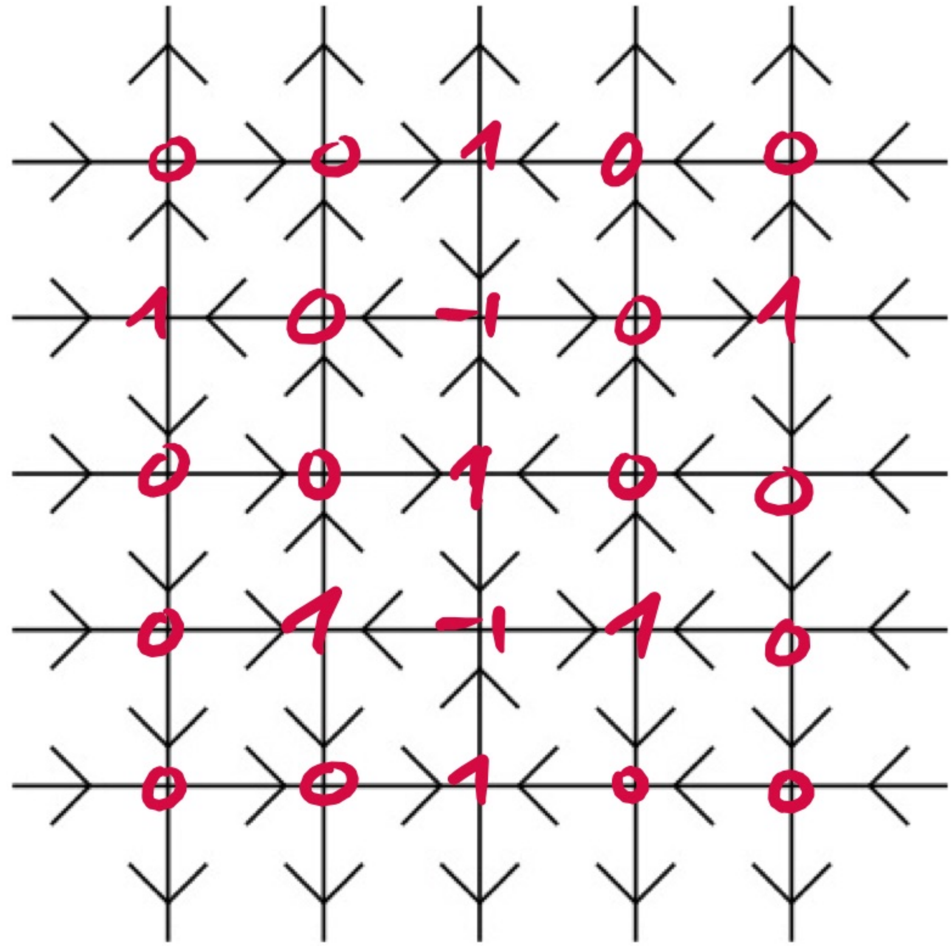
- The ASMs express the Laurent property of C.A.
  $A_{i,j} \in \{0, \pm 1\}$ + alternance condition along rows and columns $\rightarrow$ 0..010..0-10..010..0-10..010-0
- generalize permutation matrices
- λ-determinant (here with $\lambda = 1$)

Equivalence with 6 vertex model and DWBC

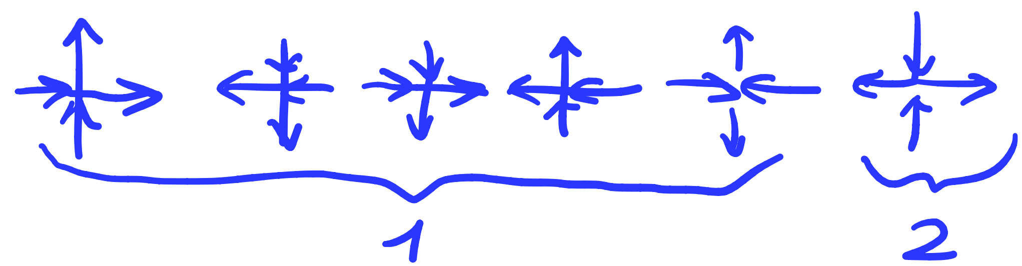
ASM

≡

6 Vertex



T = 2-enumeration



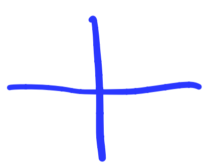
More questions:

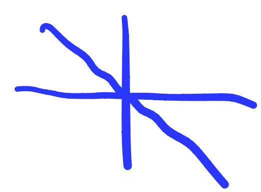
- general 6 Vertex ? (away from dimer point)
- other models ?

More answers:

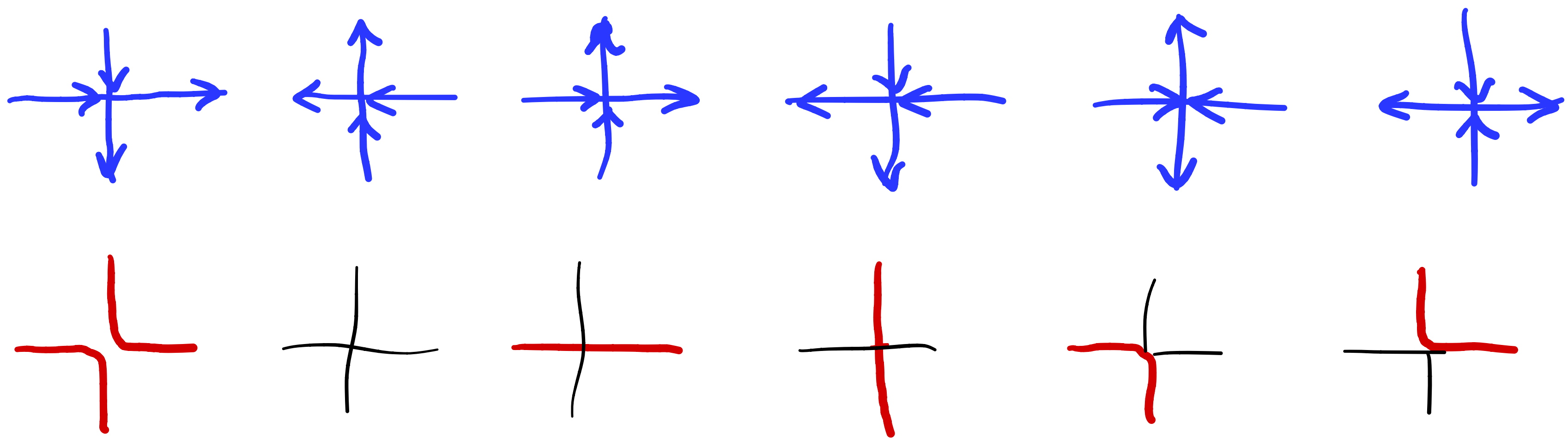
- The tangent method applies to interacting paths

e.g. 6 Vertex , 20 Vertex w/ integrable weights

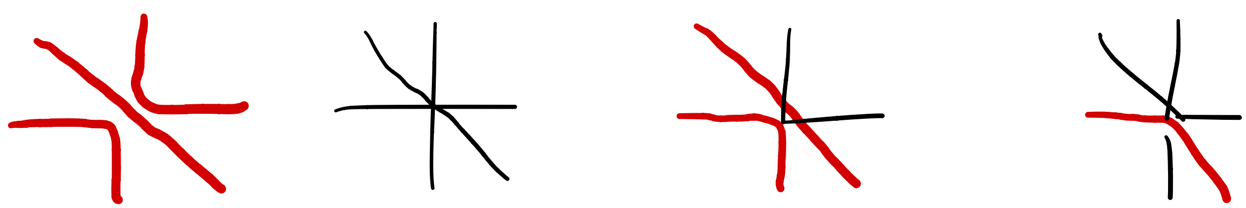




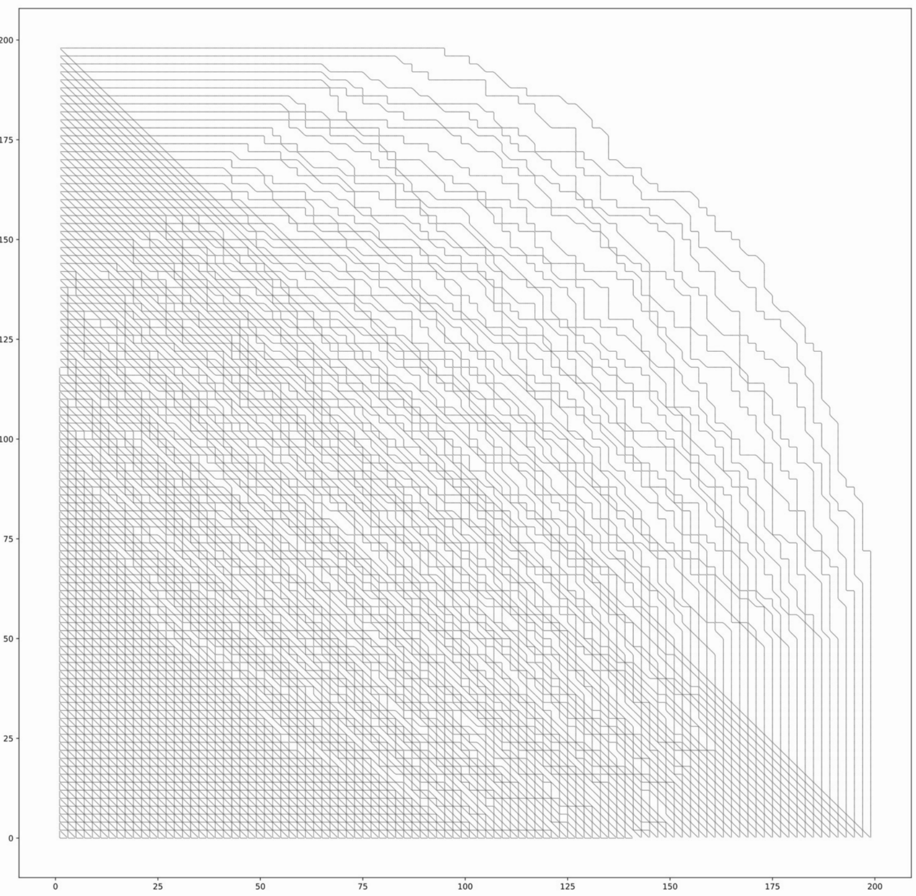
6V model = osculating paths



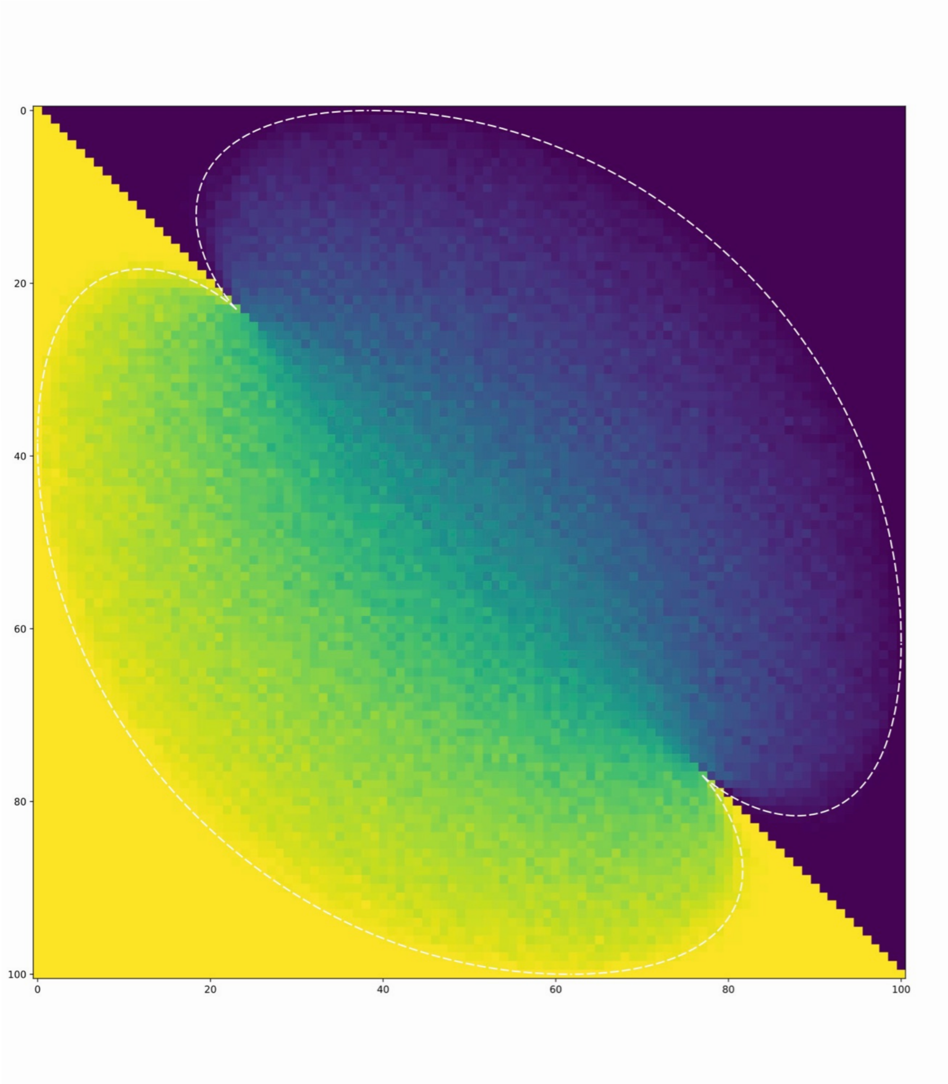
20V model = osculating Schröder paths



1 1 9 9



N=200



N=100

20V model = "osculating schröder paths"

To Conclude

cluster algebra $\longleftrightarrow$ limit shapes

networks

Non-intersecting paths

tangent method

?

osculating paths

ASM = 1 particular ex of laurent phenomenon

generalize ?

Thank

You!