Expressive curves

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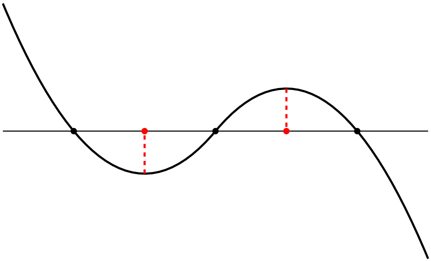
arXiv:2006.14066 (with E. Shustin)

arXiv:1711.10598 (with P. Pylyavskyy, E. Shustin, D. Thurston)

Theorem

Let $g(x) \in \mathbb{R}[x]$ be a polynomial whose roots are real and distinct. Then g has exactly one critical point between each pair of consecutive roots, and no other critical points (even over $\mathbb{C}$).



Expressive curves

$G(x,y) \in \mathbb{R}[x,y] \subset \mathbb{C}[x,y]$ polynomial with real coefficients
$C = \{(x,y) \in \mathbb{C}^2 \mid G(x,y) = 0\}$ affine plane algebraic curve
$C_{\mathbb{R}} = \{(x,y) \in \mathbb{R}^2 \mid G(x,y) = 0\}$ set of real points of C

Definition

Polynomial G (resp., curve C) is called *expressive* if

- all critical points of G are real;
- at each critical point, G has a nondegenerate Hessian;
- each bounded connected component of $\mathbb{R}^2 \setminus C_{\mathbb{R}}$ contains exactly one critical point of G;
- each unbounded component of $\mathbb{R}^2 \setminus C_{\mathbb{R}}$ contains no critical points;
- $C_{\mathbb{R}}$ is connected, and contains infinitely many points.

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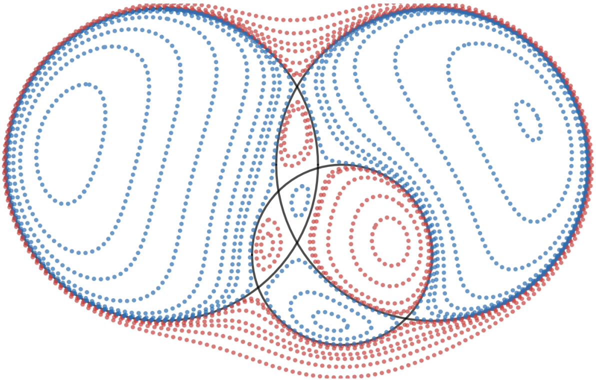
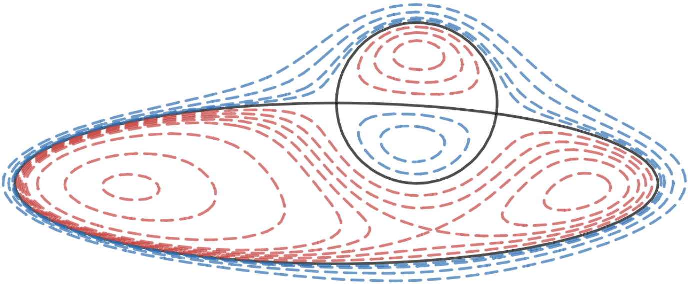
Example of an expressive curve



Example of a non-expressive curve



Motivations

Our main result is a complete classification of expressive curves (subject to a mild technical condition).

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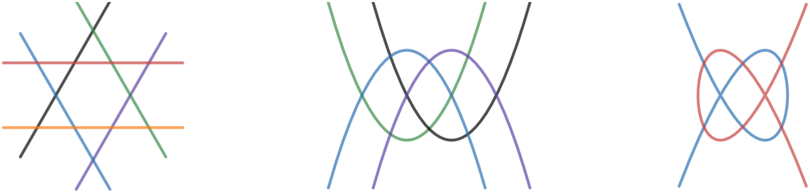
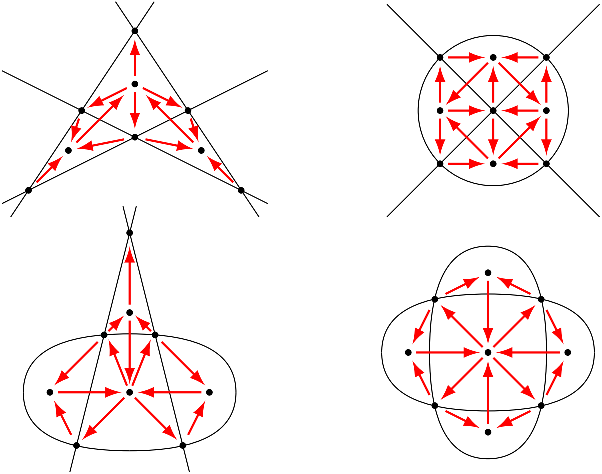
Motivation #1: Extending the theory of hyperplane arrangements



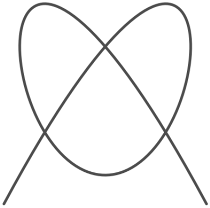
From plane curves to cluster theory

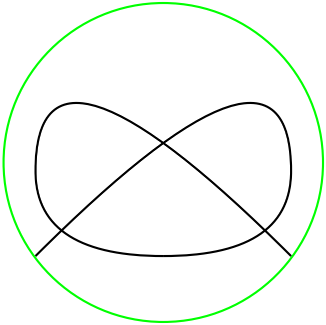
Motivation #2: Understanding the geometry and topology of plane curves using combinatorics of quiver mutations and plabic graphs



Curve → divide

A nodal curve in the real affine plane defines a *divide*.





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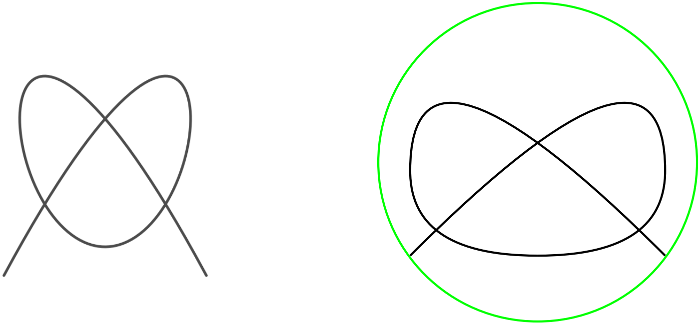

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Divide → plabic graph

Plabic (planar bicolored) graphs were introduced by A. Postnikov to study parametrizations of cells in totally nonnegative Grassmannians. All our plabic graphs are *trivalent-univalent*.

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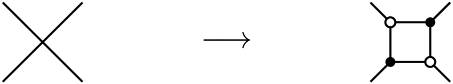

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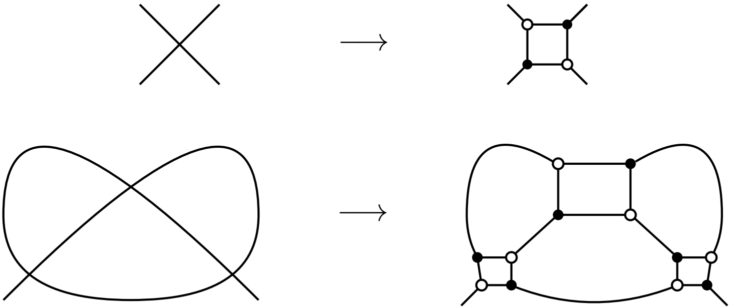
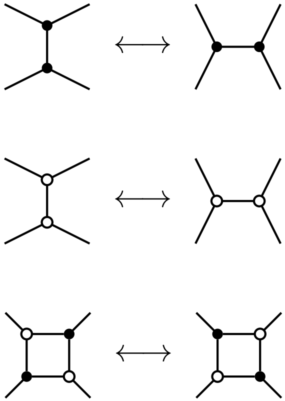


Move equivalence of plabic graphs

Two plabic graphs are called *move equivalent* if they can be obtained from each other via repeated application of the following moves:

flip moves

square move



Plabic graph → quiver

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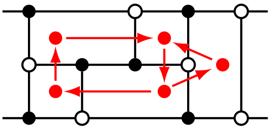 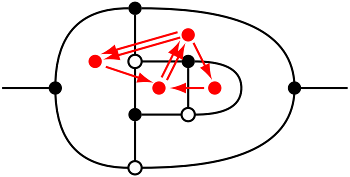

Square moves on plabic graphs translate into quiver mutations:

Flip moves do not change the quiver.

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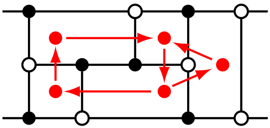

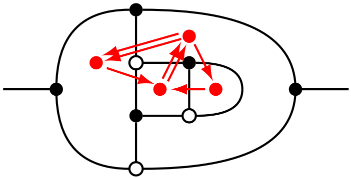

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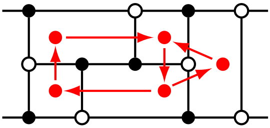

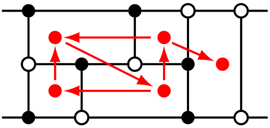

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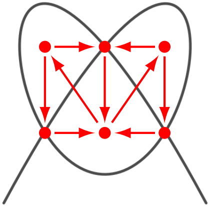
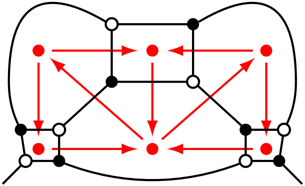
Curve → divide → plabic graph → quiver

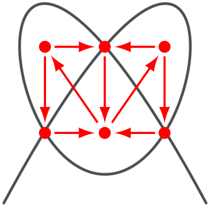


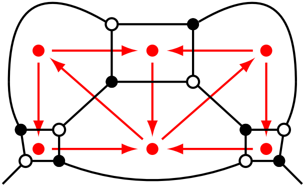


Conjecture

Two plabic graphs coming from expressive curves are move equivalent if and only if their quivers are mutation equivalent.

Curve → divide → plabic graph → quiver





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Curve → divide → plabic graph → link

There is a construction [T. Kawamura + FPST] that associates a canonical (transverse) link to any plabic graph.

Theorem (SF-P. Pylyavskyy-E. Shustin-D. Thurston)

The link of a plabic graph is invariant under local moves.

Theorem (N. A'Campo + FPST)

The link of a divide arising from a real morsification of a plane curve singularity is isotopic to the link of the singularity.

We conjecture that under mild technical assumptions, the link of a divide arising from an expressive curve is isotopic to the curve's *link at infinity.*

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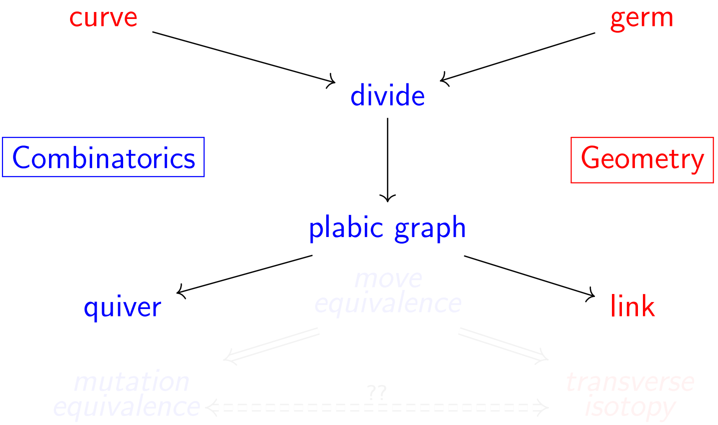
Mutation equivalence *vs.* link equivalence



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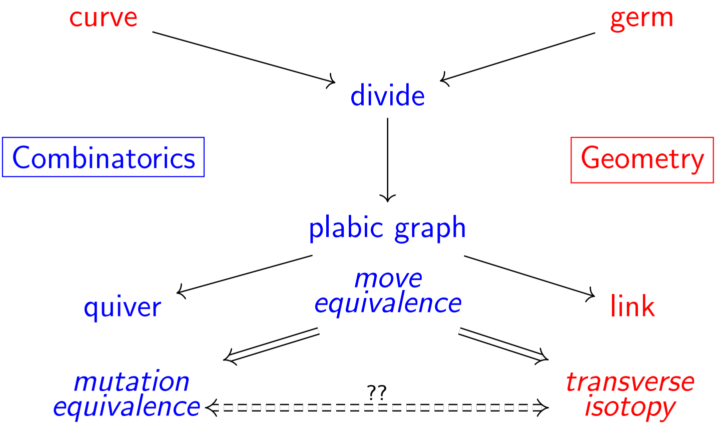
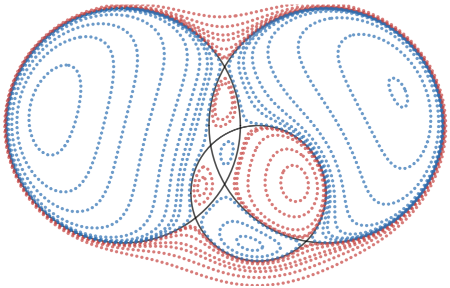


Back to expressive curves

A real plane algebraic curve C is *expressive* if its defining polynomial has the smallest number of critical points allowed by the topology of the set of real points of C.



L^∞-regular curves

x, y, z	projective coordinates in $\mathbb{P}^2$
$L^\infty = \{z = 0\}$	line at infinity
$\mathbb{C}^2 = \mathbb{P}^2 \setminus L^\infty$	affine complex plane

Definition

A projective curve $C = Z(F) \subset \mathbb{P}^2$ is called *L^∞-regular* if

$$\forall p \in C \cap L^\infty \quad (Z(\tfrac{\partial F}{\partial x}) \cdot Z(\tfrac{\partial F}{\partial y}))_p = \mu(C, p) + (C \cdot L^\infty)_p - 1.$$

An affine curve $C \subset \mathbb{C}^2$ is called *L^∞-regular* if its projective closure $\widehat{C} \subset \mathbb{P}$ is L^∞-regular.

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Polynomial and trigonometric curves

Definition

A real rational curve $C \subset \mathbb{C}^2$ is called *polynomial* if it admits a real polynomial parametrization $t \mapsto (P(t), Q(t))$.

Proposition

C is polynomial $\Leftrightarrow$ C is a real rational curve with one place at infinity.

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A real rational curve $C \subset \mathbb{C}^2$ is called *trigonometric* if $C_{\mathbb{R}}$ admits a real trigonometric parametrization $t \mapsto (P(\cos t, \sin t), Q(\cos t, \sin t))$.

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Expressivity criterion

Theorem

Let $C \subset \mathbb{C}^2$ be a reduced real algebraic curve, with all irreducible components real. The following are equivalent:

- *C is expressive and L^∞-regular;*
- *each component of C is either trigonometric or polynomial, all singular points of C in the affine plane are hyperbolic nodes, and the set of real points of C in the affine plane is connected.*

In particular, any polynomial or trigonometric curve all of whose singular points (away from infinity) are real hyperbolic nodes is both expressive and L^∞-regular.

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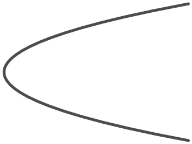
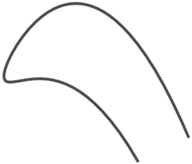
Constructing expressive curves

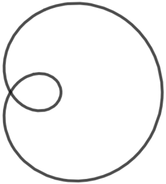
We describe many procedures for constructing new expressive curves from existing examples.



$4y^2-3y-x=0$



$4(x^2+y)^2-3(x^2+y)-x=0$



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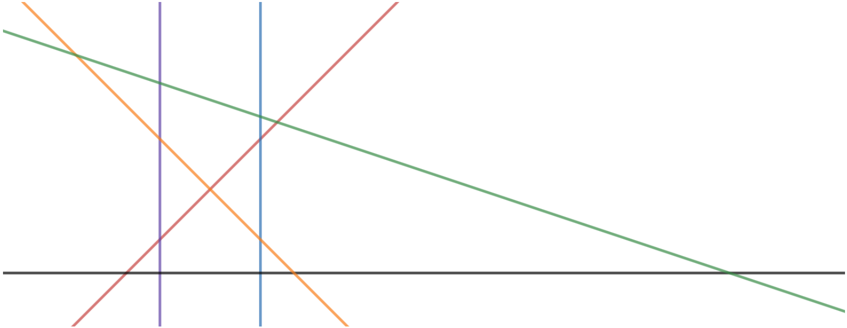
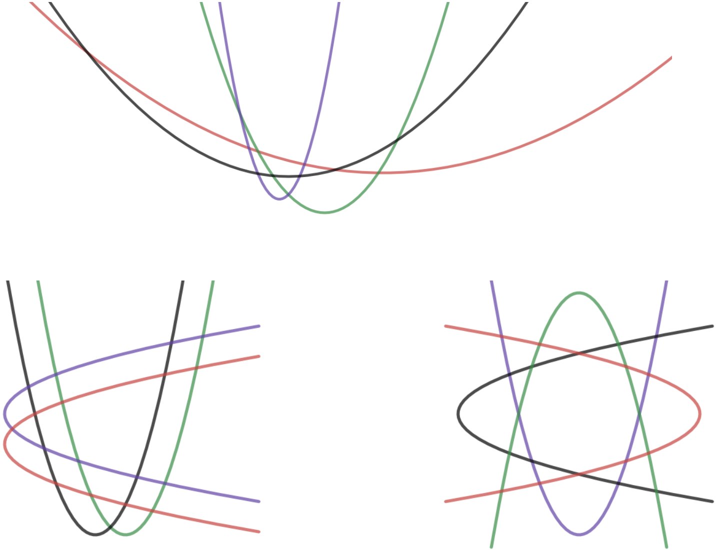
Example I: Line arrangements

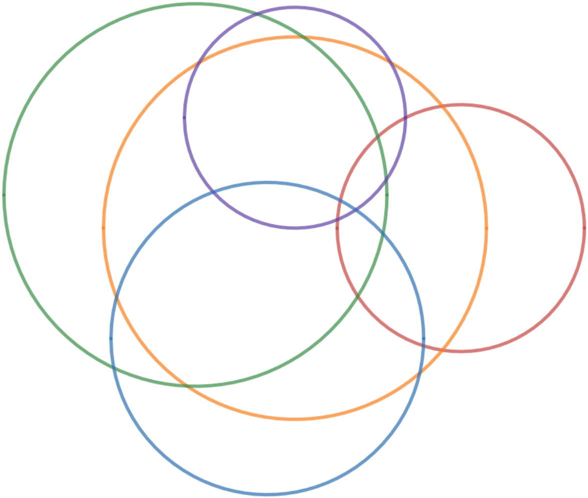


A nodal connected real line arrangement is an expressive curve.

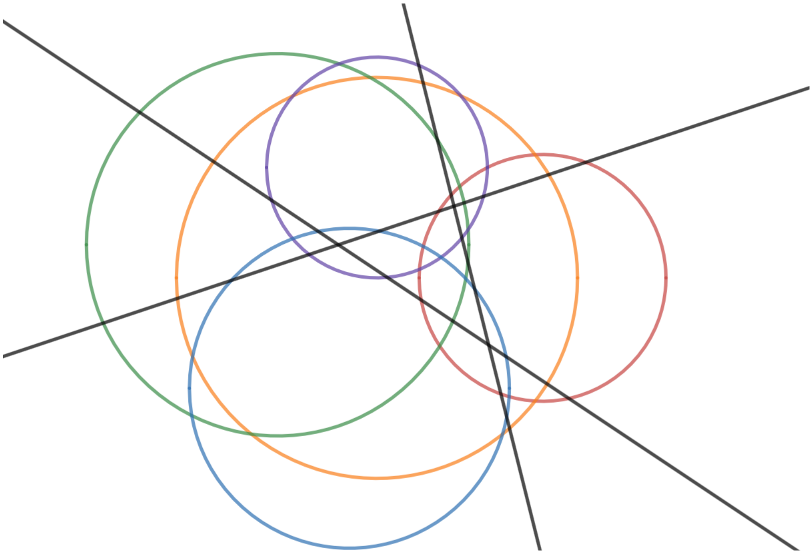
Example II: Arrangements of parabolas



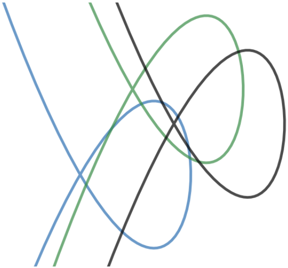
Example III: Circle arrangements

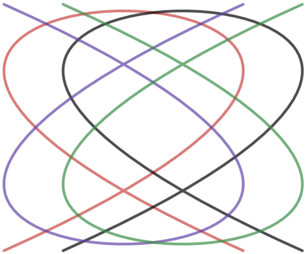


Example IV: Arrangements of lines and circles

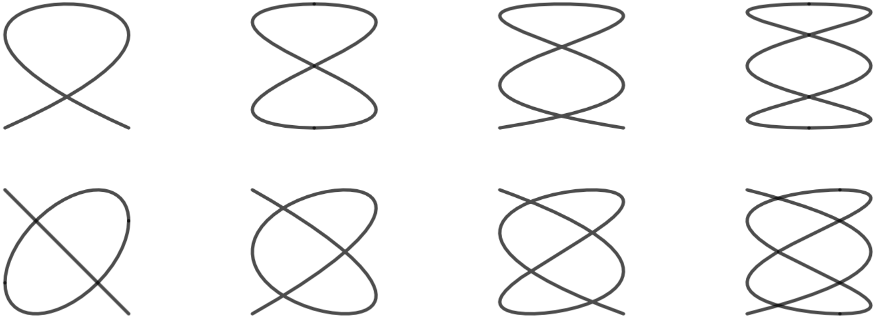


Example V: Arrangements of nodal cubics

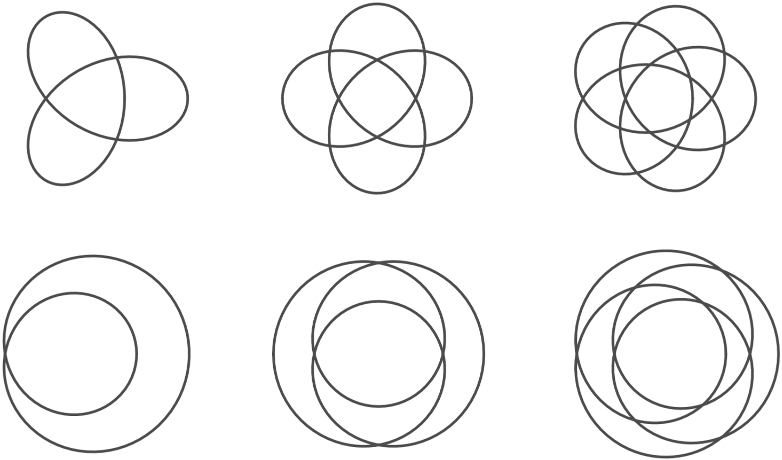




Example VI: Lissajous-Chebyshev curves



Example VII: Hypotrochoids and epitrochoids



Pseudoline arrangements

A *pseudoline arrangement* is a connected divide whose branches are embedded intervals, and any two of them intersect at most once.

Proposition

A pseudoline arrangement comes from a morsification of an isolated plane curve singularity iff any two pseudolines in it intersect.

Proposition

A pseudoline arrangement comes from an expressive L^∞-regular curve (with all components real) iff it is stretchable.

Thus there are divides which come from morsifications but not from expressive curves, or vice versa.

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