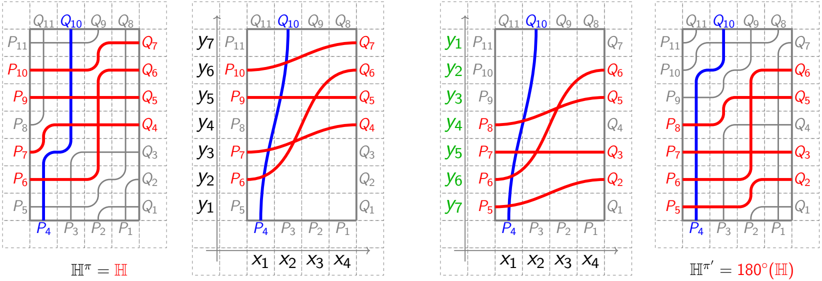
Symmetries of stochastic colored vertex models

Pavel Galashin (UCLA)

Dimers in Combinatorics and Cluster Algebras 2020

arXiv:2003.06330



$\mathbb{H}^{\pi} = \mathbb{H}$ $\mathbb{V}^{\pi} = \mathbb{V}$

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

$\mathbb{H}^{\pi'} = 180^\circ(\mathbb{H})$ $\mathbb{V}^{\pi'} = \mathbb{V}$

Stochastic colored six-vertex model

- Introduced in 2016:

[KMMO16] A. Kuniba, V. V. Mangazeev, S. Maruyama, and M. Okado. Stochastic R matrix for $U_q(A_n^{(1)})$. *Nuclear Phys. B*, 913:248–277, 2016.

Stochastic colored six-vertex model

- Introduced in 2016:

[KMMO16] A. Kuniba, V. V. Mangazeev, S. Maruyama, and M. Okado. Stochastic R matrix for $U_q(A_n^{(1)})$. *Nuclear Phys. B*, 913:248–277, 2016.

- Limiting cases include many other interesting probabilistic models

Stochastic colored six-vertex model

- Introduced in 2016:

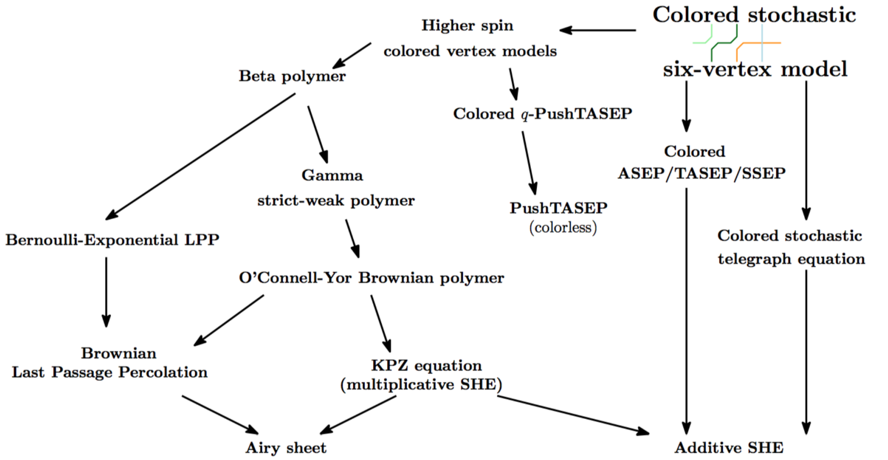
[KMMO16] A. Kuniba, V. V. Mangazeev, S. Maruyama, and M. Okado. Stochastic R matrix for $U_q(A_n^{(1)})$. *Nuclear Phys. B*, 913:248–277, 2016.

- Limiting cases include many other interesting probabilistic models

Stochastic colored six-vertex model

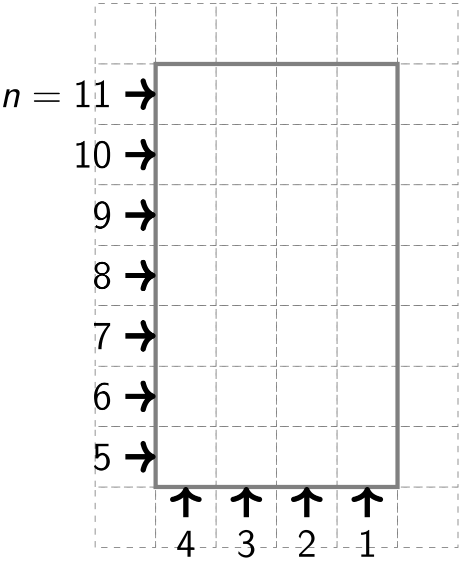
- Introd
[KMMO ... ochastic
R

- Limiti

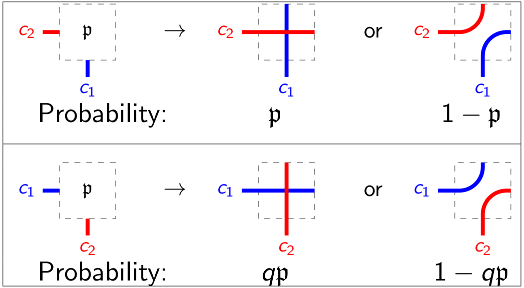


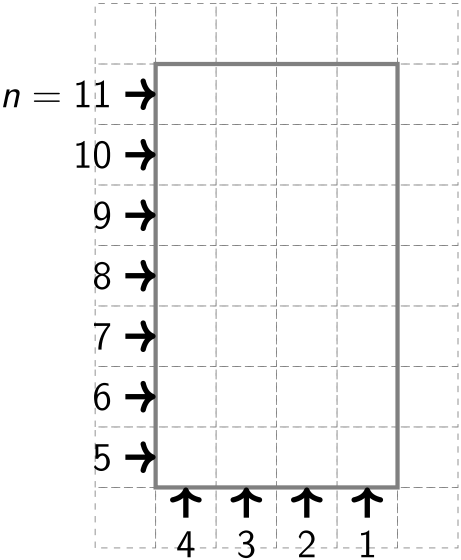
[BGW19] Alexei Borodin, Vadim Gorin, and Michael Wheeler. Shift-invariance for vertex models and polymers. arXiv:1912.02957, 2019.

- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$

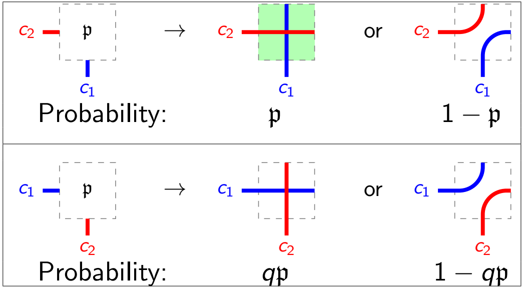


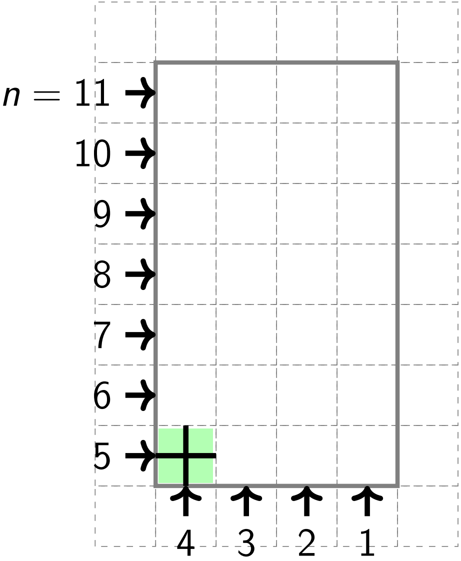
- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow



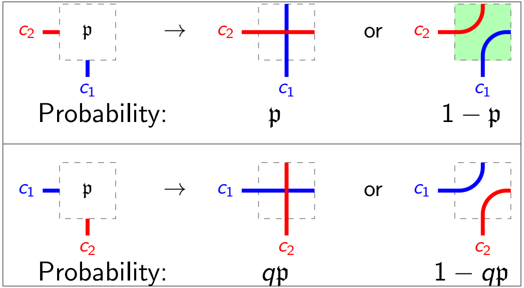


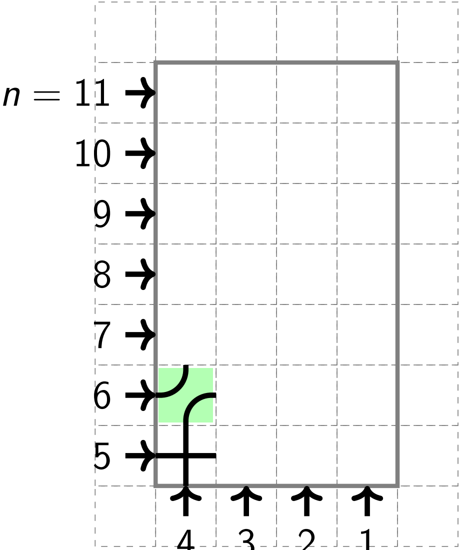
- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow



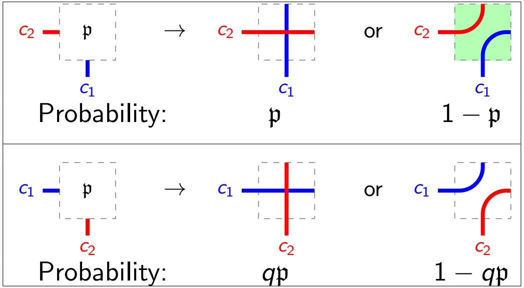


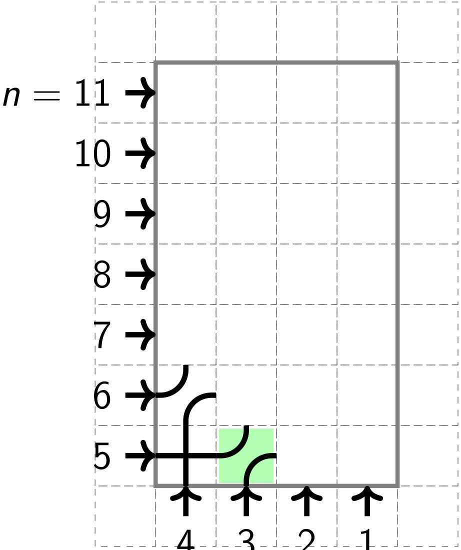
- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow



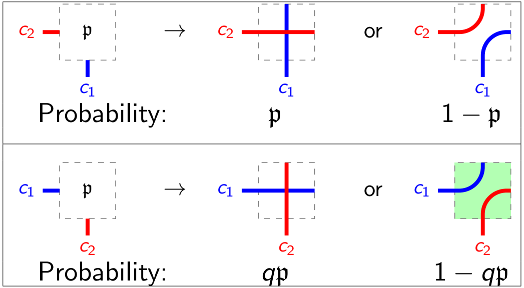


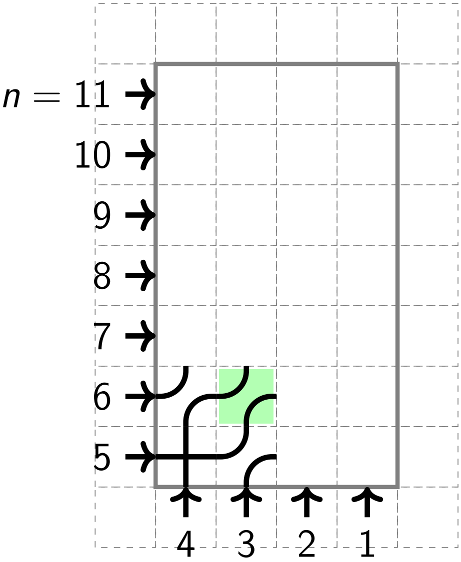
- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow



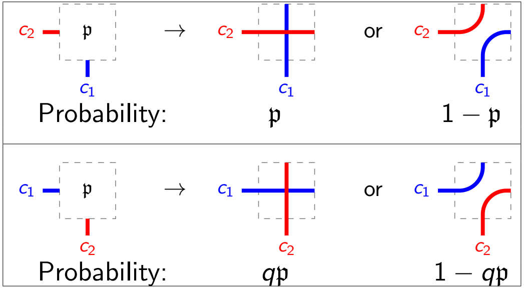


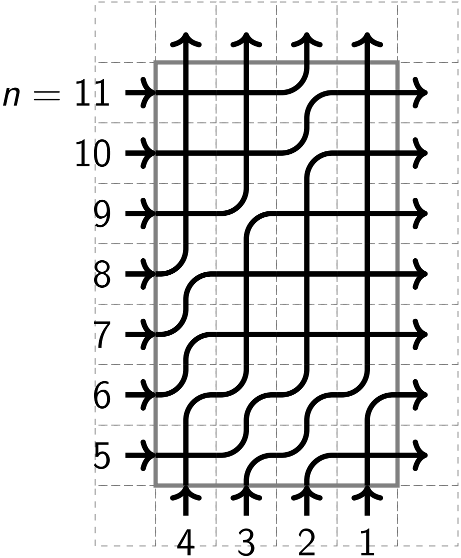
- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow



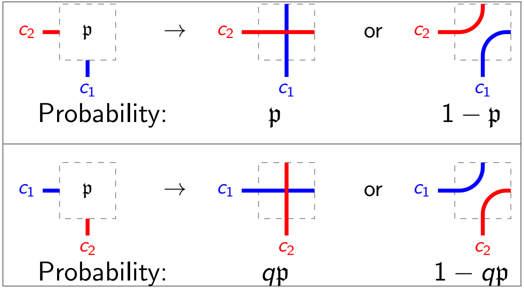


- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow

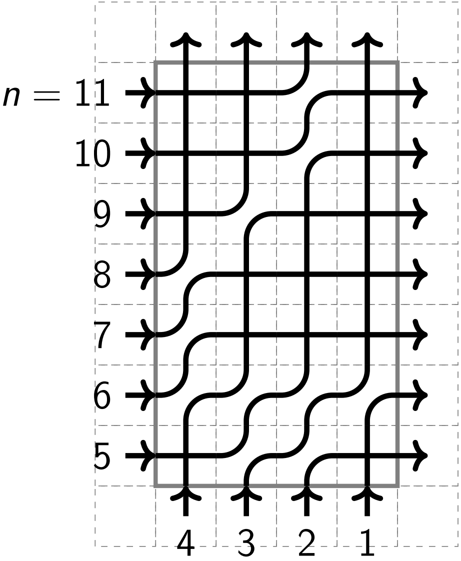




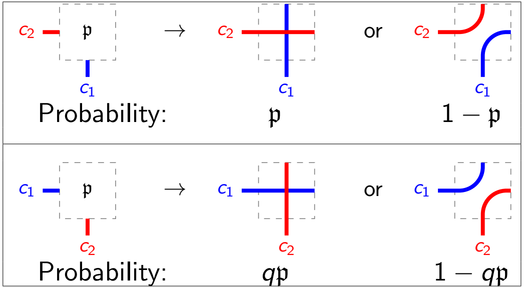
- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow



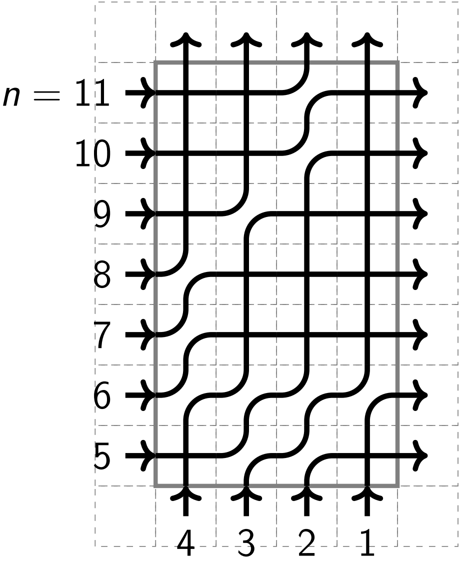
- $0 < q < 1$ is fixed



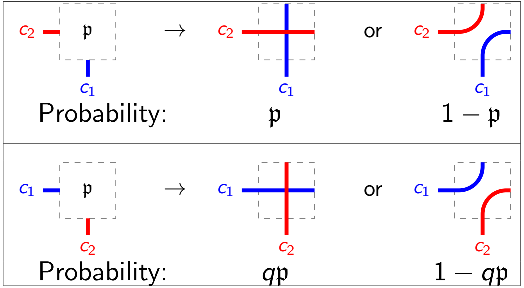
- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow



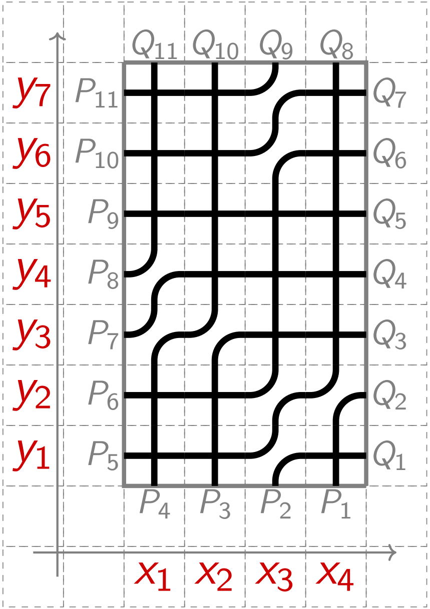
- $0 < q < 1$ is fixed
- spectral parameter $\mathfrak{p}$ depends on the square

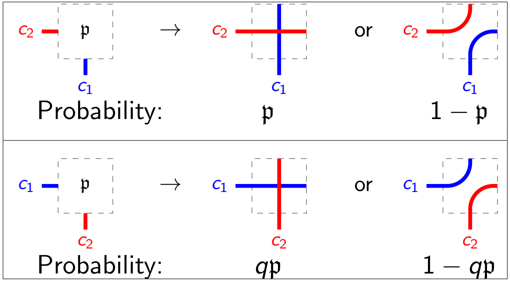


- n lattice paths of colors $1, 2, \ldots, n$ move up/right on $\mathbb{Z}^2$
- When two paths of colors $c_1 < c_2$ enter a square from the bottom/left, they form either a crossing or an elbow

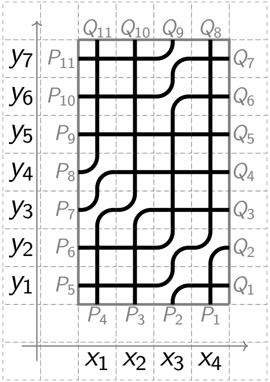


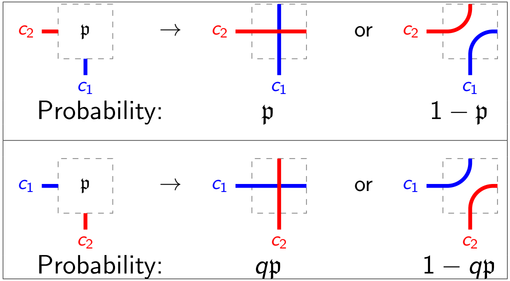
- $0 < q < 1$ is fixed
- spectral parameter $\mathfrak{p}$ depends on the square
- $\mathfrak{p}_{i,j} = \dfrac{y_j - x_i}{y_j - qx_i}$



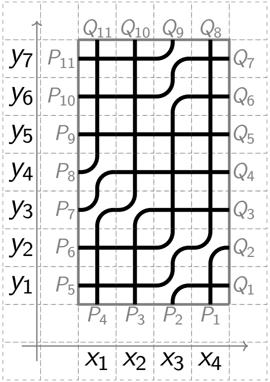


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$

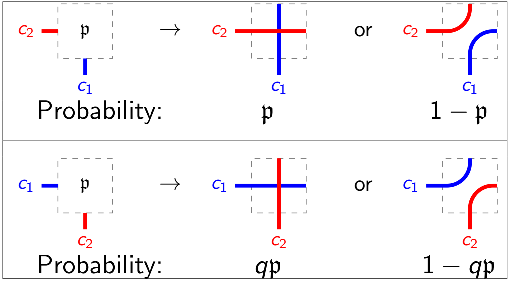


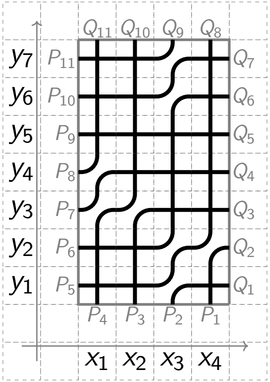


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- 2^{MN} pipe dreams $\longrightarrow$ $n!$ permutations



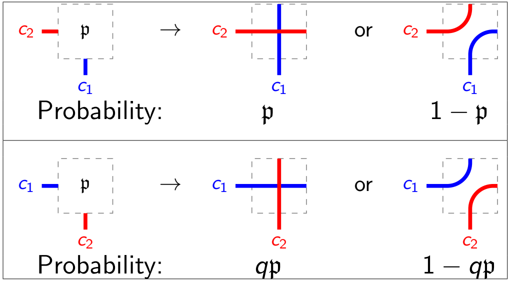
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 10 & 8 & 6 & 4 & 11 & 5 & 7 & 9 \end{pmatrix}$$



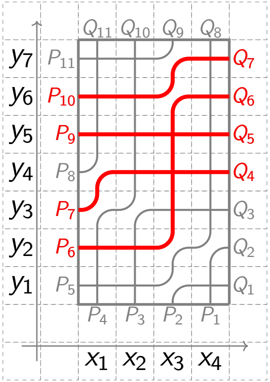


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - qx_i}$
- 2^{MN} pipe dreams $\longrightarrow$ $n!$ permutations
- For each $\pi \in S_n$, let $\mathbb{H}^\pi$ and $\mathbb{V}^\pi$ record the endpoints of all "horizontal" and "vertical" pipes

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 10 & 8 & 6 & 4 & 11 & 5 & 7 & 9 \end{pmatrix}$$

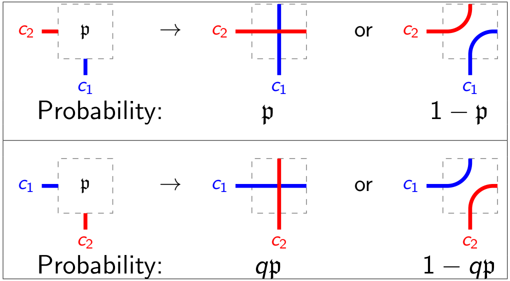


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- 2^{MN} pipe dreams $\longrightarrow$ $n!$ permutations
- For each $\pi \in S_n$, let $\mathbb{H}^\pi$ and $\mathbb{V}^\pi$ record the endpoints of all "horizontal" and "vertical" pipes

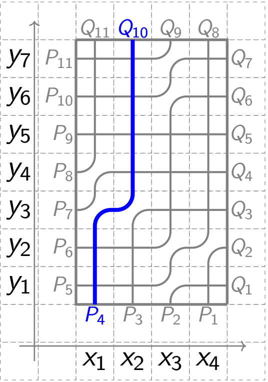


$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 10 & 8 & 6 & 4 & 11 & 5 & 7 & 9 \end{pmatrix}$$

$$\mathbb{H}^\pi = \{(6,6), (7,4), (9,5), (10,7)\}$$



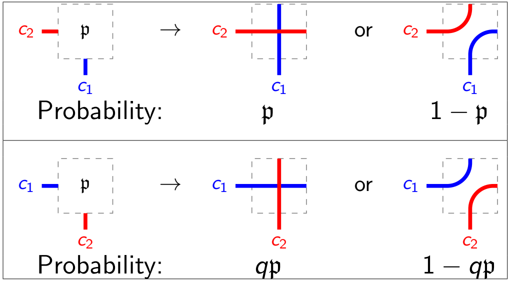
- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - qx_i}$
- 2^{MN} pipe dreams $\longrightarrow$ $n!$ permutations
- For each $\pi \in S_n$, let $\mathbb{H}^\pi$ and $\mathbb{V}^\pi$ record the endpoints of all "horizontal" and "vertical" pipes



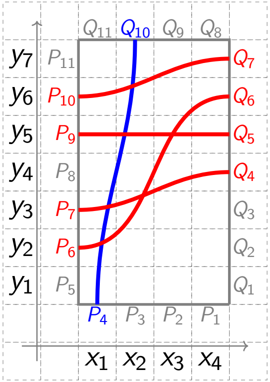
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 10 & 8 & 6 & 4 & 11 & 5 & 7 & 9 \end{pmatrix}$$

$$\mathbb{H}^\pi = \{(6,6),(7,4),(9,5),(10,7)\}$$

$$\mathbb{V}^\pi = \{(4,10)\}$$



- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- 2^{MN} pipe dreams $\longrightarrow n!$ permutations
- For each $\pi \in S_n$, let $\mathbb{H}^\pi$ and $\mathbb{V}^\pi$ record the endpoints of all "horizontal" and "vertical" pipes
- Given $\mathbb{H}, \mathbb{V}$, let $Z^{\mathbb{H},\mathbb{V}}(\mathbf{x}, \mathbf{y})$ =probability of observing $\pi \in S_n$ with $\mathbb{H}^\pi = \mathbb{H}$ and $\mathbb{V}^\pi = \mathbb{V}$.

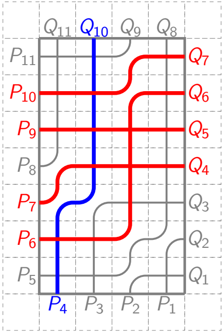


$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 10 & 8 & 6 & 4 & 11 & 5 & 7 & 9 \end{pmatrix}$$

$$\mathbb{H}^\pi = \{(6,6),(7,4),(9,5),(10,7)\}$$

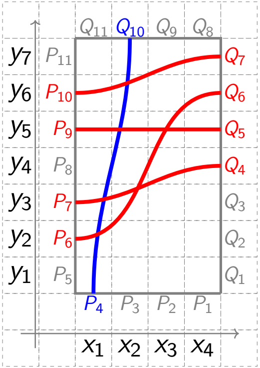
$$\mathbb{V}^\pi = \{(4,10)\}$$

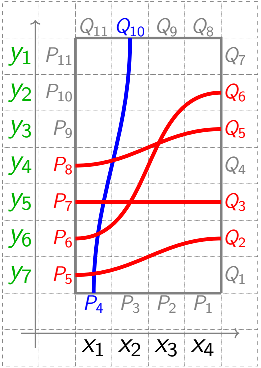
Flip theorem (G., 2020)



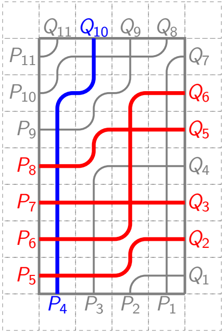
$\mathbb{H}^{\pi} = \mathbb{H}$

$\mathbb{V}^{\pi} = \mathbb{V}$



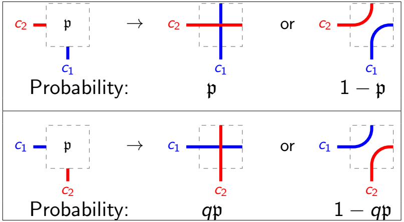


$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$



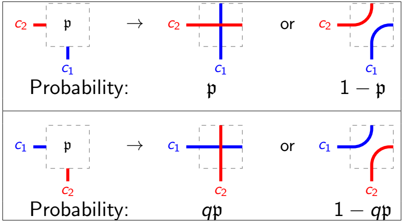
$\mathbb{H}^{\pi'} = 180^\circ(\mathbb{H})$

$\mathbb{V}^{\pi'} = \mathbb{V}$



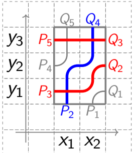
- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - qx_i}$
- Flip theorem:

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

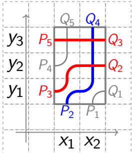


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- Flip theorem:

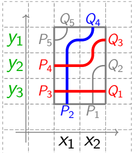
$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$



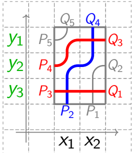
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$1-q\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$



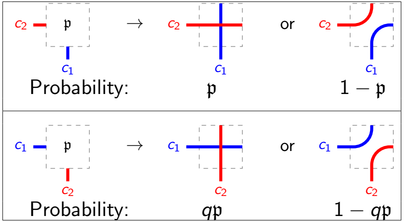
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$\mathfrak{p}_{2,2}$
$1-\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$



$1-\mathfrak{p}_{1,1}$	$1-q\mathfrak{p}_{2,1}$
$\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

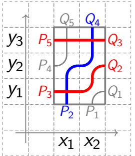


$1-\mathfrak{p}_{1,1}$	$\mathfrak{p}_{2,1}$
$1-\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

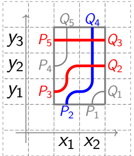


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- Flip theorem:

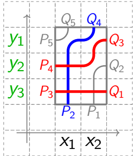
$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$



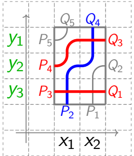
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$1-q\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$



$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$\mathfrak{p}_{2,2}$
$1-\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$

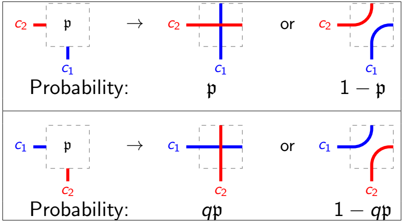


$1-\mathfrak{p}_{1,1}$	$1-q\mathfrak{p}_{2,1}$
$\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$



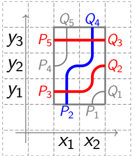
$1-\mathfrak{p}_{1,1}$	$\mathfrak{p}_{2,1}$
$1-\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

- All 4 probabilities are different!

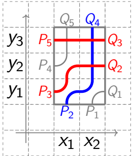


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- Flip theorem:

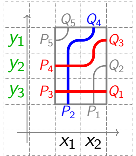
$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$



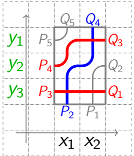
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$1-q\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$



$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$\mathfrak{p}_{2,2}$
$1-\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$

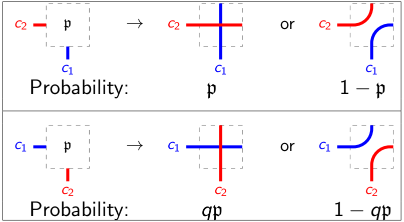


$1-\mathfrak{p}_{1,1}$	$1-q\mathfrak{p}_{2,1}$
$\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$



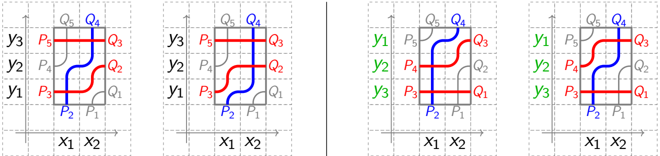
$1-\mathfrak{p}_{1,1}$	$\mathfrak{p}_{2,1}$
$1-\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

- All 4 probabilities are different!
- $\Longrightarrow$ there is no weight-preserving bijection



- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- Flip theorem:

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$



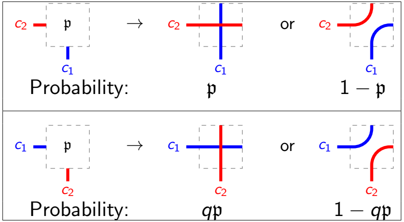
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$1-q\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$

$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$\mathfrak{p}_{2,2}$
$1-\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$

$1-\mathfrak{p}_{1,1}$	$1-q\mathfrak{p}_{2,1}$
$\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

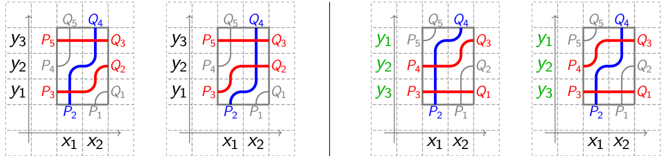
$1-\mathfrak{p}_{1,1}$	$\mathfrak{p}_{2,1}$
$1-\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

- All 4 probabilities are different!
- $\Longrightarrow$ there is no weight-preserving bijection
- Sum of the first two $=$ sum of the second two



- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$
- Flip theorem:

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\text{rev}(\mathbf{y}))$$



$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$1-q\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$

$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$\mathfrak{p}_{2,2}$
$1-\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$

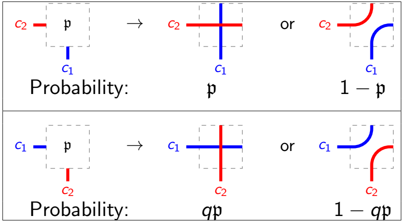
$1-\mathfrak{p}_{1,1}$	$1-q\mathfrak{p}_{2,1}$
$\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

$1-\mathfrak{p}_{1,1}$	$\mathfrak{p}_{2,1}$
$1-\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

- All 4 probabilities are different!
- $\Longrightarrow$ there is no weight-preserving bijection
- Sum of the first two = sum of the second two

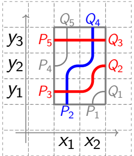
Corollary

The number of pipe dreams for $Z^{\mathbb{H},\mathbb{V}}$ equals the number of pipe dreams for $Z^{180^\circ(\mathbb{H}),\mathbb{V}}$.

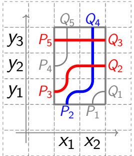


- $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - qx_i}$
- Flip theorem:

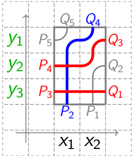
$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$



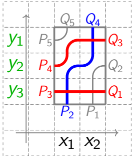
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$1-q\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$



$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$
$1-\mathfrak{p}_{1,2}$	$\mathfrak{p}_{2,2}$
$1-\mathfrak{p}_{1,1}$	$1-\mathfrak{p}_{2,1}$



$1-\mathfrak{p}_{1,1}$	$1-q\mathfrak{p}_{2,1}$
$\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$



$1-\mathfrak{p}_{1,1}$	$\mathfrak{p}_{2,1}$
$1-\mathfrak{p}_{1,2}$	$1-\mathfrak{p}_{2,2}$
$\mathfrak{p}_{1,3}$	$\mathfrak{p}_{2,3}$

- All 4 probabilities are different!
- $\Longrightarrow$ there is no weight-preserving bijection
- Sum of the first two $=$ sum of the second two

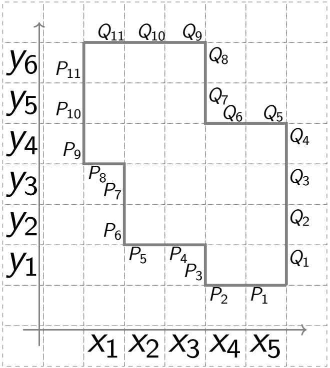
Corollary

The number of pipe dreams for $Z^{\mathbb{H},\mathbb{V}}$ *equals the number of pipe dreams for* $Z^{180^\circ(\mathbb{H}),\mathbb{V}}$.

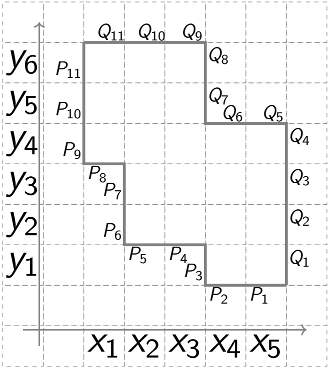
Problem

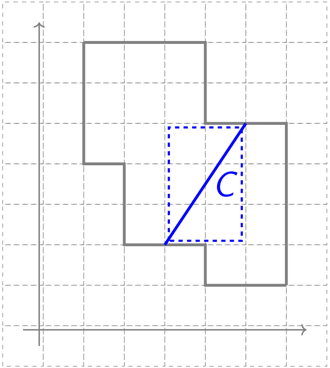
Find a bijective proof.

Generalized flip theorem



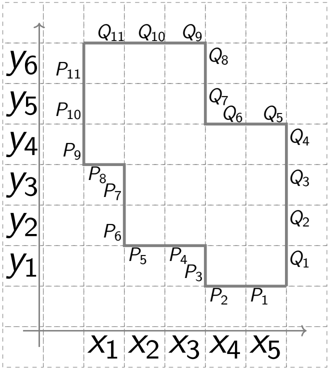
Generalized flip theorem

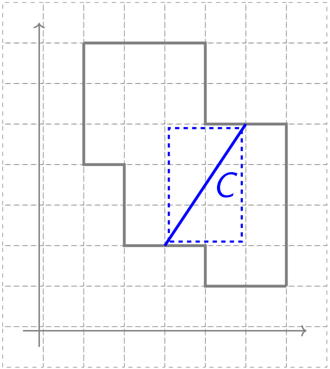


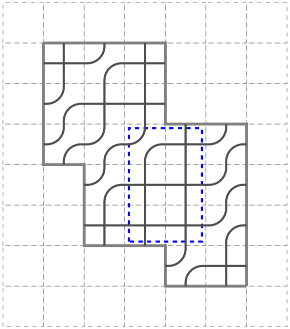


- Cut C

Generalized flip theorem



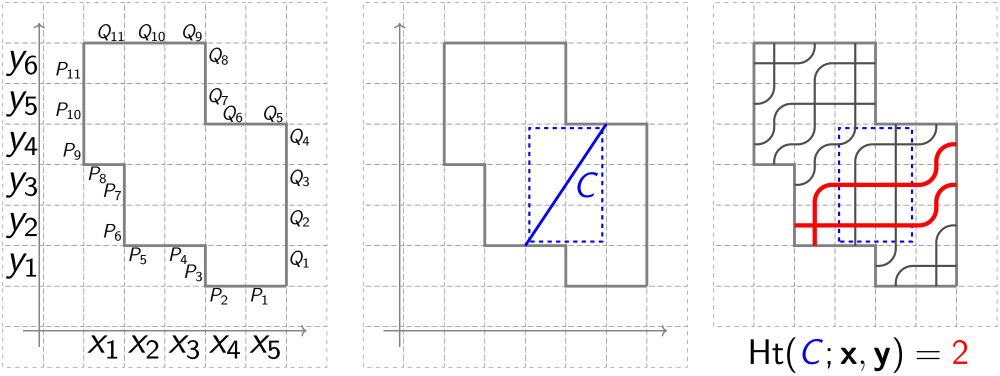




$\mathrm{Ht}(C; \mathbf{x}, \mathbf{y}) =$

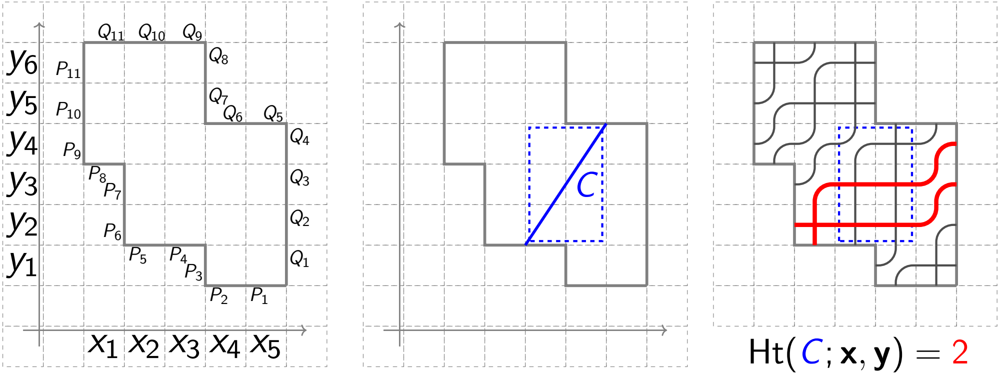
- Cut C $\mapsto$ random variable $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$

Generalized flip theorem



- Cut $C \mapsto$ random variable $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$
- $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$ measures the number of pipes that "cross" C from left to right

Generalized flip theorem

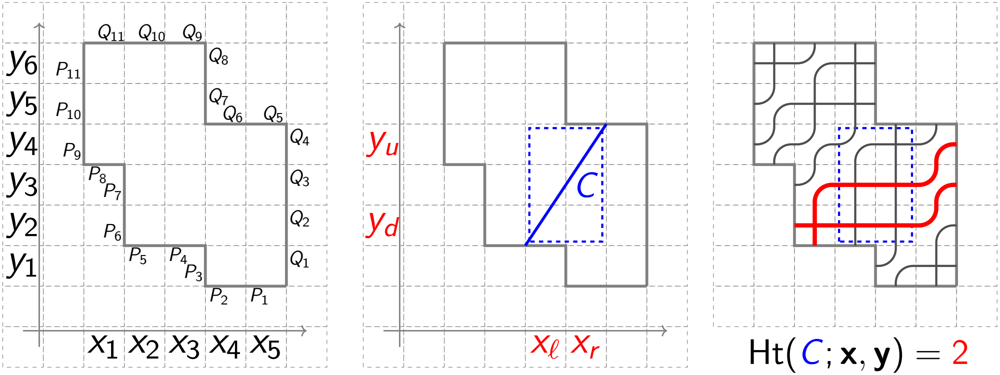


$\mathrm{Ht}(C; \mathbf{x}, \mathbf{y}) = 2$

- Cut $C \mapsto$ random variable $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$
- $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$ measures the number of pipes that "cross" C from left to right
- Fact: distribution of $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$ is a symmetric function in

$\mathrm{supp}_H(C; \mathbf{x}) := \{x_\ell, x_{\ell+1}, \dots, x_r\}$ and $\mathrm{supp}_V(C; \mathbf{y}) := \{y_d, y_{d+1}, \dots, y_u\}$

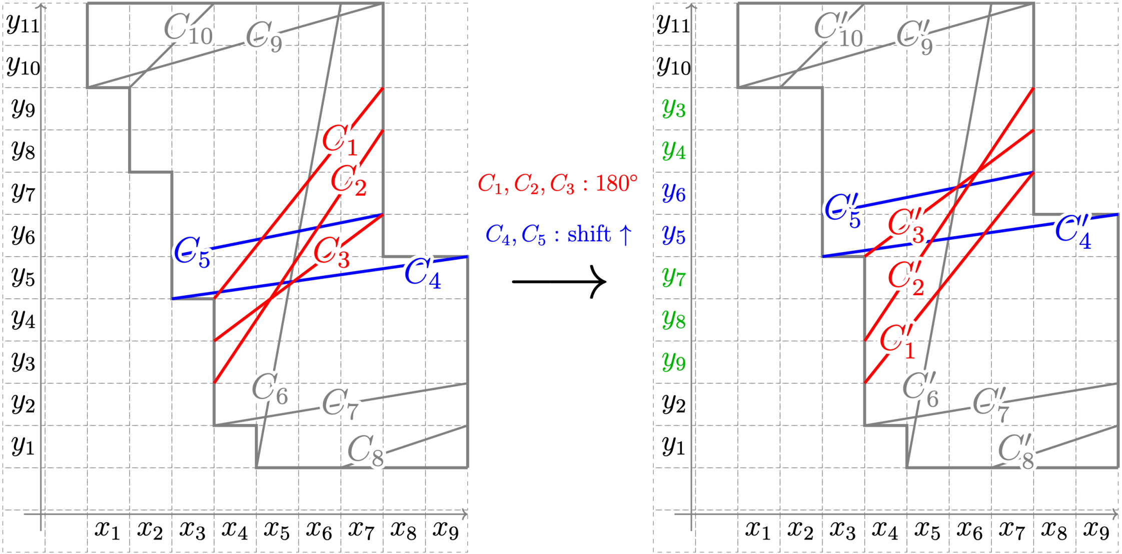
Generalized flip theorem



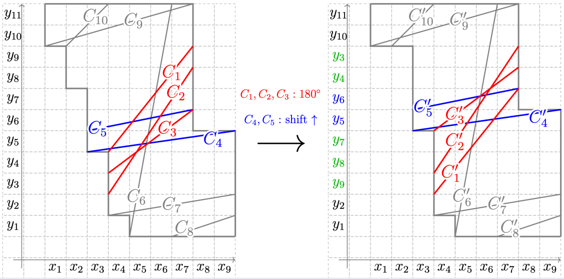
- Cut $C \mapsto$ random variable $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$
- $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$ measures the number of pipes that "cross" C from left to right
- Fact: distribution of $\mathrm{Ht}(C; \mathbf{x}, \mathbf{y})$ is a symmetric function in

$\mathrm{supp}_H(C; \mathbf{x}) := \{x_\ell, x_{\ell+1}, \dots, x_r\}$ and $\mathrm{supp}_V(C; \mathbf{y}) := \{y_d, y_{d+1}, \dots, y_u\}$

Theorem 2 (G., 2020)

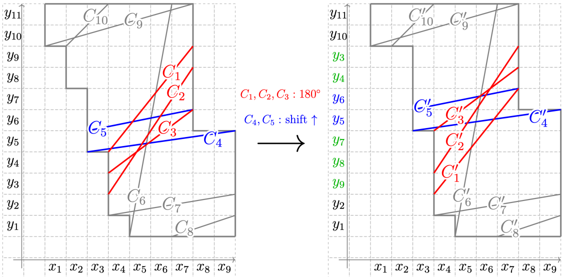


Theorem 2 (G., 2020)



Given any cuts $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_m$, we have

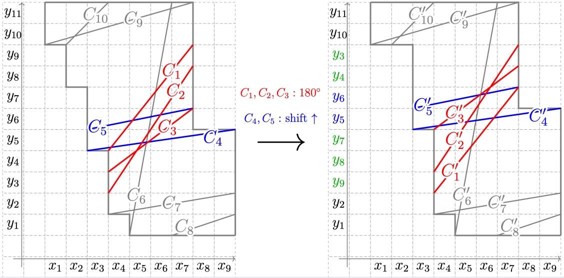
Theorem 2 (G., 2020)



Given any cuts $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_m$, we have

$$\mathrm{supp}_H(C_i; \mathbf{x}) = \mathrm{supp}_H(C'_i; \mathbf{x}') \quad \text{and} \quad \mathrm{supp}_V(C_i; \mathbf{y}) = \mathrm{supp}_V(C'_i; \mathbf{y}') \quad \text{for all } i$$

Theorem 2 (G., 2020)

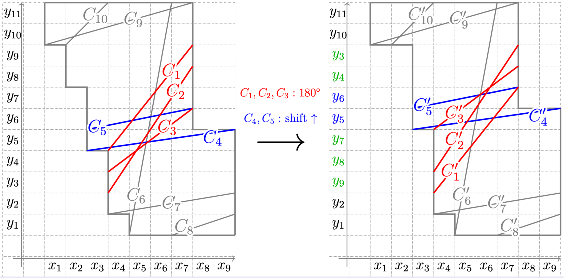


Given any cuts $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_m$, we have

$$\text{supp}_H(C_i; \mathbf{x}) = \text{supp}_H(C'_i; \mathbf{x}') \quad \text{and} \quad \text{supp}_V(C_i; \mathbf{y}) = \text{supp}_V(C'_i; \mathbf{y}') \quad \text{for all } i$$

if and only if

Theorem 2 (G., 2020)



Given any cuts $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_m$, we have

$$\mathrm{supp}_H(C_i; \mathbf{x}) = \mathrm{supp}_H(C'_i; \mathbf{x}') \quad \text{and} \quad \mathrm{supp}_V(C_i; \mathbf{y}) = \mathrm{supp}_V(C'_i; \mathbf{y}') \quad \text{for all } i$$

if and only if

$$\Big(\mathrm{Ht}(C_1; \mathbf{x}, \mathbf{y}), \ldots, \mathrm{Ht}(C_m; \mathbf{x}, \mathbf{y})\Big) \stackrel{d}{=} \Big(\mathrm{Ht}(C'_1; \mathbf{x}', \mathbf{y}'), \ldots, \mathrm{Ht}(C'_m; \mathbf{x}', \mathbf{y}')\Big).$$

Theorem 2 (G., 2020)

- In other words, if a transformation preserves individual distributions of $\mathrm{Ht}(C_i; \mathbf{x}, \mathbf{y})$-s then it preserves their joint distribution.

Given any cuts $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_m$, we have

$$\mathrm{supp}_H(C_i; \mathbf{x}) = \mathrm{supp}_H(C'_i; \mathbf{x}') \quad \text{and} \quad \mathrm{supp}_V(C_i; \mathbf{y}) = \mathrm{supp}_V(C'_i; \mathbf{y}') \quad \text{for all } i$$

if and only if

$$\Big(\mathrm{Ht}(C_1; \mathbf{x}, \mathbf{y}), \ldots, \mathrm{Ht}(C_m; \mathbf{x}, \mathbf{y})\Big) \stackrel{d}{=} \Big(\mathrm{Ht}(C'_1; \mathbf{x}', \mathbf{y}'), \ldots, \mathrm{Ht}(C'_m; \mathbf{x}', \mathbf{y}')\Big).$$

Theorem 2 (G., 2020)

- In other words, if a transformation preserves individual distributions of $\mathrm{Ht}(C_i; \mathbf{x}, \mathbf{y})$-s then it preserves their joint distribution.
- Unexpected behavior – these random variables are far from independent!

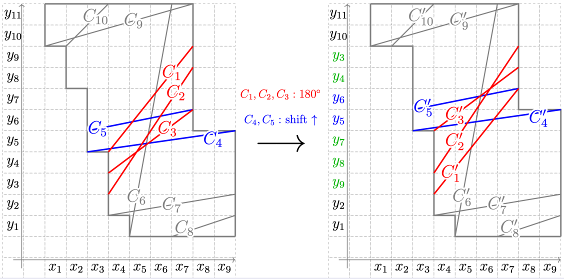
Given any cuts $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_m$, we have

$$\mathrm{supp}_H(C_i; \mathbf{x}) = \mathrm{supp}_H(C'_i; \mathbf{x}') \quad \text{and} \quad \mathrm{supp}_V(C_i; \mathbf{y}) = \mathrm{supp}_V(C'_i; \mathbf{y}') \quad \text{for all } i$$

if and only if

$$\Big(\mathrm{Ht}(C_1; \mathbf{x}, \mathbf{y}), \ldots, \mathrm{Ht}(C_m; \mathbf{x}, \mathbf{y})\Big) \stackrel{d}{=} \Big(\mathrm{Ht}(C'_1; \mathbf{x}', \mathbf{y}'), \ldots, \mathrm{Ht}(C'_m; \mathbf{x}', \mathbf{y}')\Big).$$

Theorem 2 (G., 2020)



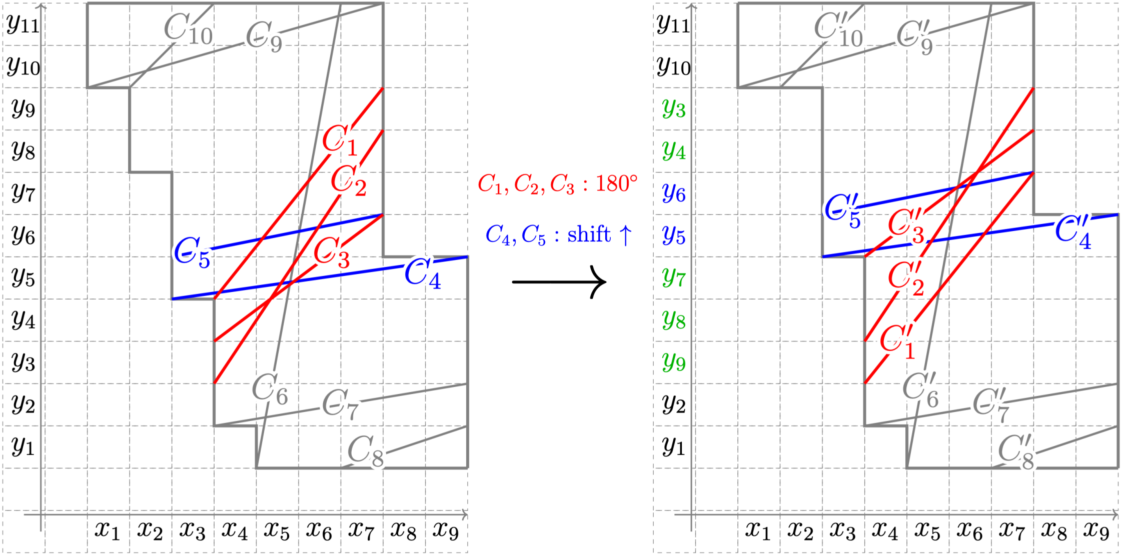
Given any cuts $C_1, \ldots, C_m$ and $C'_1, \ldots, C'_m$, we have

$$\mathrm{supp}_H(C_i; \mathbf{x}) = \mathrm{supp}_H(C'_i; \mathbf{x}') \quad \text{and} \quad \mathrm{supp}_V(C_i; \mathbf{y}) = \mathrm{supp}_V(C'_i; \mathbf{y}') \quad \text{for all } i$$

if and only if

$$\Big(\mathrm{Ht}(C_1; \mathbf{x}, \mathbf{y}), \ldots, \mathrm{Ht}(C_m; \mathbf{x}, \mathbf{y})\Big) \overset{d}{=} \Big(\mathrm{Ht}(C'_1; \mathbf{x}', \mathbf{y}'), \ldots, \mathrm{Ht}(C'_m; \mathbf{x}', \mathbf{y}')\Big).$$

Theorem 2 (G., 2020)



Theorem 1

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

Theorem 2

$$\mathrm{supp}_H(C_i;\mathbf{x}) = \mathrm{supp}_H(C_i';\mathbf{x}')$$
$$\mathrm{supp}_V(C_i;\mathbf{y}) = \mathrm{supp}_V(C_i';\mathbf{y}')$$

if and only if

$$\Big(\mathrm{Ht}(C_1;\mathbf{x},\mathbf{y}),\dots,\mathrm{Ht}(C_m;\mathbf{x},\mathbf{y})\Big) \stackrel{d}{=} \Big(\mathrm{Ht}(C_1';\mathbf{x}',\mathbf{y}'),\dots,\mathrm{Ht}(C_m';\mathbf{x}',\mathbf{y}')\Big)$$

Theorem 1

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

Theorem 2

$$\mathrm{supp}_H(C_i;\mathbf{x}) = \mathrm{supp}_H(C_i';\mathbf{x}')$$
$$\mathrm{supp}_V(C_i;\mathbf{y}) = \mathrm{supp}_V(C_i';\mathbf{y}')$$

if and only if

$$\Big(\mathrm{Ht}(C_1;\mathbf{x},\mathbf{y}),\dots,\mathrm{Ht}(C_m;\mathbf{x},\mathbf{y})\Big) \overset{d}{=} \Big(\mathrm{Ht}(C_1';\mathbf{x}',\mathbf{y}'),\dots,\mathrm{Ht}(C_m';\mathbf{x}',\mathbf{y}')\Big)$$

- Theorem 2 $\implies$ Theorem 1

Theorem 1

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

Theorem 2

$$\mathrm{supp}_H(C_i;\mathbf{x}) = \mathrm{supp}_H(C_i';\mathbf{x}')$$
$$\mathrm{supp}_V(C_i;\mathbf{y}) = \mathrm{supp}_V(C_i';\mathbf{y}')$$

if and only if

$$\Big(\mathrm{Ht}(C_1;\mathbf{x},\mathbf{y}),\ldots,\mathrm{Ht}(C_m;\mathbf{x},\mathbf{y})\Big) \stackrel{d}{=} \Big(\mathrm{Ht}(C_1';\mathbf{x}',\mathbf{y}'),\ldots,\mathrm{Ht}(C_m';\mathbf{x}',\mathbf{y}')\Big)$$

- Theorem 2 $\Longrightarrow$ Theorem 1
- Theorem 2 generalizes the results and confirms a conjecture of

[BGW19] Alexei Borodin, Vadim Gorin, and Michael Wheeler. Shift-invariance for vertex models and polymers. arXiv:1912.02957, 2019.

Theorem 1

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

Theorem 2

$$\mathrm{supp}_H(C_i;\mathbf{x}) = \mathrm{supp}_H(C_i';\mathbf{x}')$$
$$\mathrm{supp}_V(C_i;\mathbf{y}) = \mathrm{supp}_V(C_i';\mathbf{y}')$$

if and only if

$$\Big(\mathrm{Ht}(C_1;\mathbf{x},\mathbf{y}),\dots,\mathrm{Ht}(C_m;\mathbf{x},\mathbf{y})\Big)$$
$$\stackrel{d}{=} \Big(\mathrm{Ht}(C_1';\mathbf{x}',\mathbf{y}'),\dots,\mathrm{Ht}(C_m';\mathbf{x}',\mathbf{y}')\Big)$$

- Theorem 2 $\Longrightarrow$ Theorem 1
- Theorem 2 generalizes the results and confirms a conjecture of

  [BGW19] Alexei Borodin, Vadim Gorin, and Michael Wheeler. Shift-invariance for vertex models and polymers. arXiv:1912.02957, 2019.
- Shift of [BGW19] = double flip

Theorem 1

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

Theorem 2

$$\mathrm{supp}_H(C_i;\mathbf{x}) = \mathrm{supp}_H(C_i';\mathbf{x}')$$
$$\mathrm{supp}_V(C_i;\mathbf{y}) = \mathrm{supp}_V(C_i';\mathbf{y}')$$

if and only if

$$\Big(\mathrm{Ht}(C_1;\mathbf{x},\mathbf{y}),\ldots,\mathrm{Ht}(C_m;\mathbf{x},\mathbf{y})\Big)$$
$$\overset{d}{=} \Big(\mathrm{Ht}(C_1';\mathbf{x}',\mathbf{y}'),\ldots,\mathrm{Ht}(C_m';\mathbf{x}',\mathbf{y}')\Big)$$

- Theorem 2 $\Longrightarrow$ Theorem 1
- Theorem 2 generalizes the results and confirms a conjecture of

  [BGW19] Alexei Borodin, Vadim Gorin, and Michael Wheeler. Shift-invariance for vertex models and polymers. arXiv:1912.02957, 2019.
- Shift of [BGW19] = double flip
- Proof of Theorem 1: Yang–Baxter relation combined with a Hecke algebra interpretation of the model

  [LLT97] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Flag varieties and the Yang-Baxter equation. *Lett. Math. Phys.*, 40(1):75–90, 1997.

Theorem 1

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

Theorem 2

$$\mathrm{supp}_H(C_i;\mathbf{x}) = \mathrm{supp}_H(C_i';\mathbf{x}')$$
$$\mathrm{supp}_V(C_i;\mathbf{y}) = \mathrm{supp}_V(C_i';\mathbf{y}')$$

if and only if

$$\Big(\mathrm{Ht}(C_1;\mathbf{x},\mathbf{y}),\dots,\mathrm{Ht}(C_m;\mathbf{x},\mathbf{y})\Big)$$
$$\overset{d}{=} \Big(\mathrm{Ht}(C_1';\mathbf{x}',\mathbf{y}'),\dots,\mathrm{Ht}(C_m';\mathbf{x}',\mathbf{y}')\Big)$$

- Theorem 2 $\Longrightarrow$ Theorem 1
- Theorem 2 generalizes the results and confirms a conjecture of

  [BGW19] Alexei Borodin, Vadim Gorin, and Michael Wheeler. Shift-invariance for vertex models and polymers. arXiv:1912.02957, 2019.
- Shift of [BGW19] $=$ double flip
- Proof of Theorem 1: Yang–Baxter relation combined with a Hecke algebra interpretation of the model

  [LLT97] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Flag varieties and the Yang-Baxter equation. *Lett. Math. Phys.*, 40(1):75–90, 1997.
- Hecke algebra approach also gives a one-line proof of

  [BB19] Alexei Borodin and Alexey Bufetov. Color-position symmetry in interacting particle systems. arXiv:1905.04692.

Positroid varieties

$\mathrm{Gr}(k, n) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}$

Positroid varieties

$\mathrm{Gr}(k, n) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\} = \{\text{full rank } k \times n \text{ matrices}\}/\text{row operations}.$

Positroid varieties

$\mathrm{Gr}(k,n) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\} = \{\text{full rank } k \times n \text{ matrices}\}/\text{row operations}.$

$\mathrm{Gr}(k,n)$ is stratified into positroid varieties. Here's the most interesting one:

Positroid varieties

$$\mathrm{Gr}(k,n) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\} = \{\text{full rank } k \times n \text{ matrices}\}/\text{row operations}.$$

$\mathrm{Gr}(k,n)$ is stratified into positroid varieties. Here's the most interesting one:

$$\Pi^\circ_{k,n} := \{X \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(X), \Delta_{2,\dots,k+1}(X), \dots, \Delta_{n,\dots,k-1}(X) \neq 0\},$$

where $\Delta_I(X)$ =maximal minor of X with column set I.

Positroid varieties

$$\mathrm{Gr}(k,n) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\} = \{\text{full rank } k \times n \text{ matrices}\}/\text{row operations}.$$

$\mathrm{Gr}(k,n)$ is stratified into positroid varieties. Here's the most interesting one:

$$\Pi^{\circ}_{k,n} := \{X \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(X), \Delta_{2,\dots,k+1}(X), \dots, \Delta_{n,\dots,k-1}(X) \neq 0\},$$

where $\Delta_I(X)$ =maximal minor of X with column set I.

Example

$k = 2$, $n = 4$:

$$\Pi^{\circ}_{k,n} \cong \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \middle| a \neq 0, d \neq 0, ad - bc \neq 0 \right\}.$$

Positroid varieties

$$\mathrm{Gr}(k,n) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\} = \{\text{full rank } k \times n \text{ matrices}\}/\text{row operations}.$$

$\mathrm{Gr}(k,n)$ is stratified into positroid varieties. Here's the most interesting one:

$$\Pi^\circ_{k,n} := \{X \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(X), \Delta_{2,\dots,k+1}(X), \dots, \Delta_{n,\dots,k-1}(X) \neq 0\},$$

where $\Delta_I(X)$ =maximal minor of X with column set I.

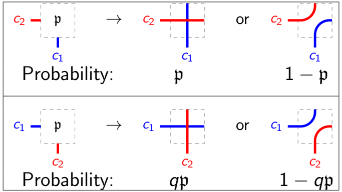
Example

$k = 2$, $n = 4$:

$$\Pi^\circ_{k,n} \cong \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \middle| a \neq 0, d \neq 0, ad - bc \neq 0 \right\}.$$

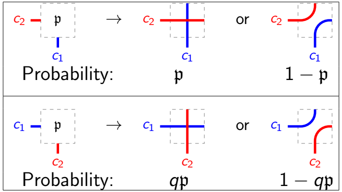
Number of such matrices over $\mathbb{F}_q$:

$$\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^2(q^2 - (q-1)) = (q-1)^4 + q(q-1)^2.$$



$\xrightarrow{\mathbf{y}\to 0}$

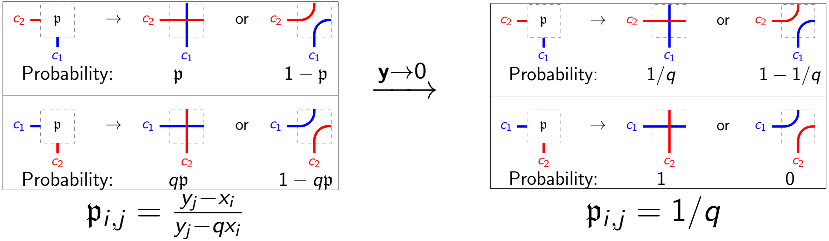
$$\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$$



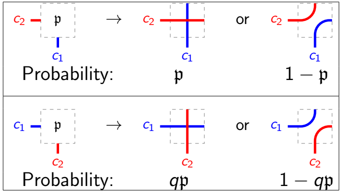
$$\xrightarrow{\mathbf{y}\to 0}$$

$$\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$$

$$\mathfrak{p}_{i,j} = 1/q$$

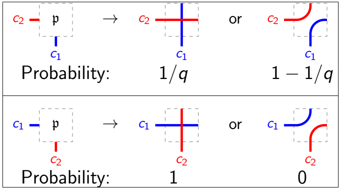


c_2 — $\mathfrak{p}$ → c_2 or c_2
c_1 c_1 c_1
Probability: $\mathfrak{p}$ $1-\mathfrak{p}$
c_1 — $\mathfrak{p}$ → c_1 or c_1
c_2 c_2 c_2
Probability: $q\mathfrak{p}$ $1-q\mathfrak{p}$
$\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - qx_i}$
$\xrightarrow{\mathbf{y}\to 0}$
c_2 — $\mathfrak{p}$ → c_2 or c_2
c_1 c_1 c_1
Probability: $1/q$ $1-1/q$
c_1 — $\mathfrak{p}$ → c_1 or c_1
c_2 c_2 c_2
Probability: 1 0
$\mathfrak{p}_{i,j} = 1/q$



$$\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$$

$$\xrightarrow{\mathbf{y} \to 0}$$

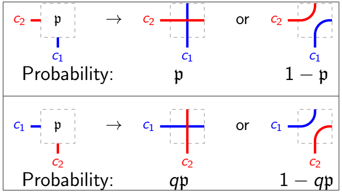


$$\mathfrak{p}_{i,j} = 1/q$$

Definition

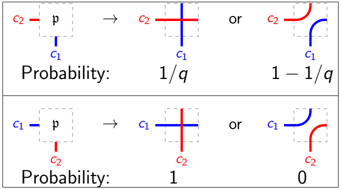
Take a $k \times (n-k)$ rectangle and let

$$Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) := \text{the probability of observing id} \in S_n \text{ when } \mathbf{y} \to 0.$$



$$\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - q x_i}$$

$$\xrightarrow{\mathbf{y}\to 0}$$



$$\mathfrak{p}_{i,j} = 1/q$$

Definition

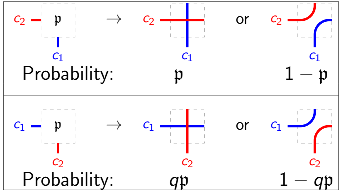
Take a $k \times (n-k)$ rectangle and let

$$Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) := \text{the probability of observing id} \in S_n \text{ when } \mathbf{y} \to 0.$$

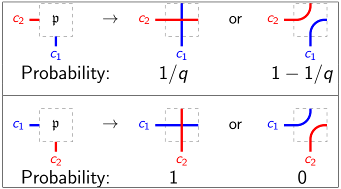
Example

$k = 2,\ n = 4$:

Probability: $(1-1/q)^4$ $\quad 1/q \cdot (1-1/q)^2$



$\xrightarrow{\mathbf{y}\to 0}$



$$\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - qx_i} \qquad \mathfrak{p}_{i,j} = 1/q$$

Definition

Take a $k \times (n-k)$ rectangle and let

$$Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) := \text{the probability of observing id} \in S_n \text{ when } \mathbf{y} \to 0.$$

Example

$k = 2,\ n = 4$:

Probability: $(1-1/q)^4$ $\quad 1/q \cdot (1-1/q)^2$

$$Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) = (1-1/q)^4 + 1/q \cdot (1-1/q)^2$$

Coincidence?

Example

$k = 2$, $n = 4$: $\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^2(q^2-(q-1)) = (q-1)^4 + q(q-1)^2.$

Coincidence?

Example

$k = 2,\ n = 4$: $\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^2(q^2 - (q-1)) = (q-1)^4 + q(q-1)^2.$

Example

$k = 2,\ n = 4$:

Probability: $(1-1/q)^4$ $\quad 1/q \cdot (1-1/q)^2$

$Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) = (1-1/q)^4 + 1/q \cdot (1-1/q)^2$

Coincidence?

Example

$k = 2,\ n = 4$: $\quad \#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^2(q^2-(q-1)) = (q-1)^4 + q(q-1)^2.$

Example

$k = 2,\ n = 4$:

Probability: $(1-1/q)^4$ $\quad$ $1/q \cdot (1-1/q)^2$

$$Z^{\mathrm{id}}_{k,n}(\mathbf{x},0) = (1-1/q)^4 + 1/q \cdot (1-1/q)^2$$

Proposition (G., 2020)

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x},0)$$

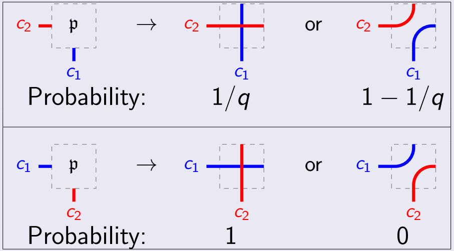
Proposition (G., 2020)

$$\#\Pi^\circ_{k,n}(\mathbb{F}_q) = q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0)$$

Proposition (G., 2020)

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0)$$

Proof.

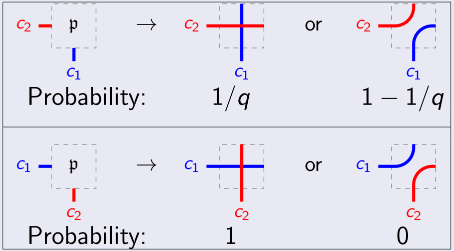


□

Proposition (G., 2020)

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0)$$

Proof.



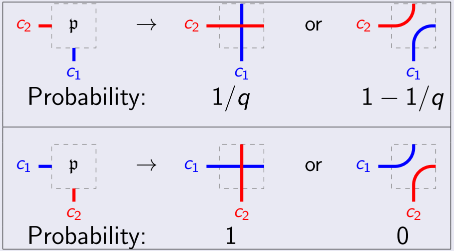
[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985.

□

Proposition (G., 2020)

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0)$$

Proof.



[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985.

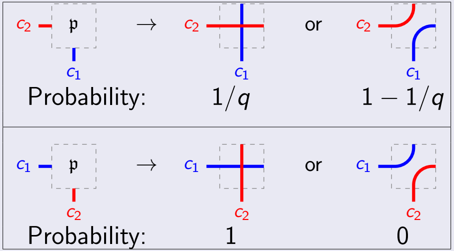
□

- $\#\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ is a Kazhdan–Lusztig R-polynomial.

Proposition (G., 2020)

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0)$$

Proof.



[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985.

□

- $\#\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ is a Kazhdan–Lusztig R-polynomial.
- The whole story generalizes to arbitrary positroid varieties.

Theorem (G.–Lam, 2020+)

Assume that $\gcd(k,n)=1$*. Then*

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot \mathrm{Cat}_{k,n}(q), \quad \textit{where} \quad \mathrm{Cat}_{k,n}(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the rational q-Catalan number.

Theorem (G.–Lam, 2020+)

Assume that $\gcd(k,n)=1$. *Then*

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot \mathrm{Cat}_{k,n}(q), \quad \textit{where} \quad \mathrm{Cat}_{k,n}(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the rational q-Catalan number.

Alternatively: the probability that a random point of $\mathrm{Gr}(k,n;\mathbb{F}_q)$ *belongs to* $\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ *is*

$$\frac{(q-1)^n}{q^n-1}.$$

Theorem (G.–Lam, 2020+)

Assume that $\gcd(k, n) = 1$. *Then*

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} \cdot \mathrm{Cat}_{k,n}(q), \quad \textit{where} \quad \mathrm{Cat}_{k,n}(q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the rational q-Catalan number.

Alternatively: the probability that a random point of $\mathrm{Gr}(k, n; \mathbb{F}_q)$ *belongs to* $\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ *is*

$$\frac{(q-1)^n}{q^n - 1}.$$

- Proof: knot theory.

Theorem (G.–Lam, 2020+)

Assume that $\gcd(k,n)=1$. *Then*

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot \mathrm{Cat}_{k,n}(q), \quad \textit{where} \quad \mathrm{Cat}_{k,n}(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the rational q-Catalan number.

Alternatively: the probability that a random point of $\mathrm{Gr}(k,n;\mathbb{F}_q)$ *belongs to* $\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ *is*

$$\frac{(q-1)^n}{q^n-1}.$$

- Proof: knot theory.
- For $q=1$, the Catalan ($n-k=k+1$) and Fuss-Catalan ($n-k=mk\pm 1$) cases are due to David Speyer.

Theorem (G.–Lam, 2020+)

Assume that $\gcd(k,n)=1$. *Then*

$$\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot \mathrm{Cat}_{k,n}(q), \quad \textit{where} \quad \mathrm{Cat}_{k,n}(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the rational q-Catalan number.

Alternatively: the probability that a random point of $\mathrm{Gr}(k,n;\mathbb{F}_q)$ *belongs to* $\Pi^\circ_{k,n}(\mathbb{F}_q)$ *is*

$$\frac{(q-1)^n}{q^n-1}.$$

- Proof: knot theory.
- For $q=1$, the Catalan $(n-k=k+1)$ and Fuss-Catalan $(n-k=mk\pm 1)$ cases are due to David Speyer.
- Forthcoming (G.–Lam, 2020+): a q,t-version.

Theorem (G.–Lam, 2020+)

Assume that $\gcd(k,n)=1$. *Then*

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q)=(q-1)^{n-1}\cdot \mathrm{Cat}_{k,n}(q), \quad \textit{where} \quad \mathrm{Cat}_{k,n}(q)=\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the rational q-Catalan number.

Alternatively: the probability that a random point of $\mathrm{Gr}(k,n;\mathbb{F}_q)$ *belongs to* $\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ *is*

$$\frac{(q-1)^n}{q^n-1}.$$

- Proof: knot theory.
- For $q=1$, the Catalan ($n-k=k+1$) and Fuss-Catalan ($n-k=mk\pm 1$) cases are due to David Speyer.
- Forthcoming (G.–Lam, 2020+): a q,t-version.

[LS16] Thomas Lam and David Speyer. Cohomology of cluster varieties. I. Locally acyclic case. arXiv:1604.06843.

Theorem (G.–Lam, 2020+)

Assume that $\gcd(k,n)=1$. *Then*

$$\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot \mathrm{Cat}_{k,n}(q), \quad \textit{where} \quad \mathrm{Cat}_{k,n}(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the rational q-Catalan number.

Alternatively: the probability that a random point of $\mathrm{Gr}(k,n;\mathbb{F}_q)$ *belongs to* $\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ *is*

$$\frac{(q-1)^n}{q^n-1}.$$

- Proof: knot theory.
- For $q=1$, the Catalan ($n-k=k+1$) and Fuss-Catalan ($n-k=mk\pm 1$) cases are due to David Speyer.
- Forthcoming (G.–Lam, 2020+): a q,t-version.

[LS16] Thomas Lam and David Speyer. Cohomology of cluster varieties. I. Locally acyclic case. arXiv:1604.06843.

[GL19] Pavel Galashin and Thomas Lam. Positroid varieties and cluster algebras. arXiv:1906.03501.

Philosophy

Stochastic colored
6-vertex model

Philosophy

Stochastic colored 6-vertex model

$\mathbf{y} \to 0$

$$q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) = \#\Pi^{\circ}_{k,n}(\mathbb{F}_q)$$

Philosophy

Stochastic colored 6-vertex model

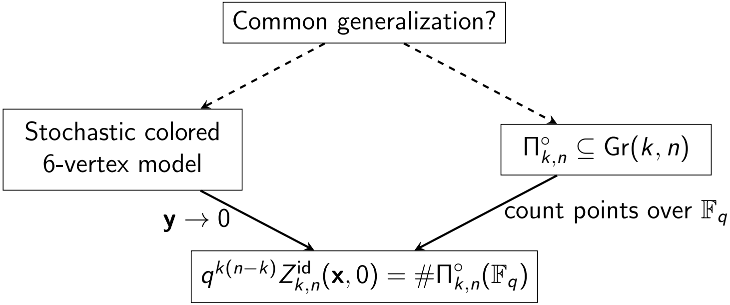
$\Pi^\circ_{k,n} \subseteq \mathrm{Gr}(k,n)$

$\mathbf{y} \to 0$

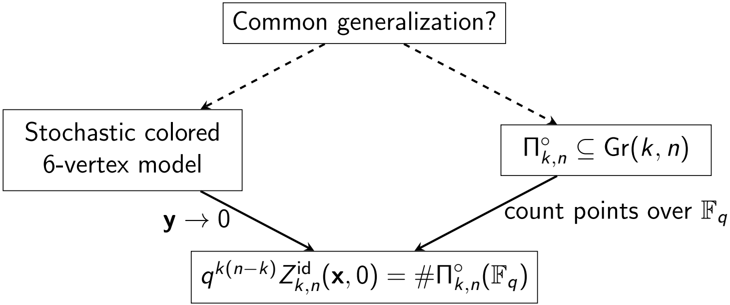
count points over $\mathbb{F}_q$

$$q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) = \#\Pi^\circ_{k,n}(\mathbb{F}_q)$$

Philosophy

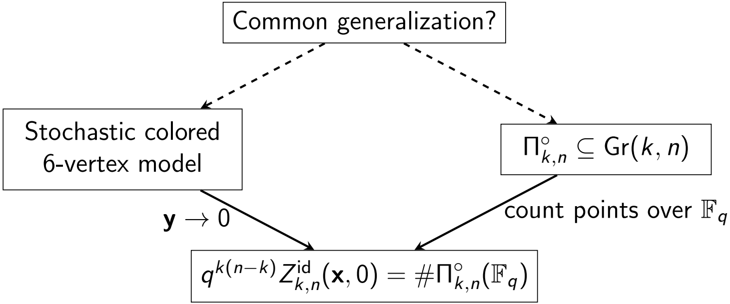

Common generalization?
Stochastic colored 6-vertex model
$\Pi^\circ_{k,n} \subseteq \mathrm{Gr}(k,n)$
$\mathbf{y} \to 0$
count points over $\mathbb{F}_q$
$q^{k(n-k)} Z^{\mathrm{id}}_{k,n}(\mathbf{x}, 0) = \#\Pi^\circ_{k,n}(\mathbb{F}_q)$

Philosophy



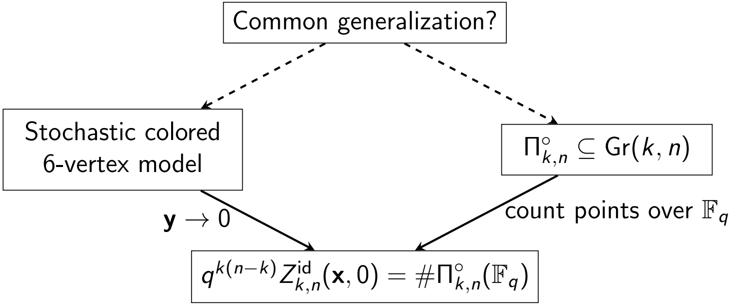
- Flip theorem as $\mathbf{y} \to 0$: $Z^{\mathbb{H},\mathbb{V}}(\mathbf{x}, 0) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x}, 0)$

Philosophy



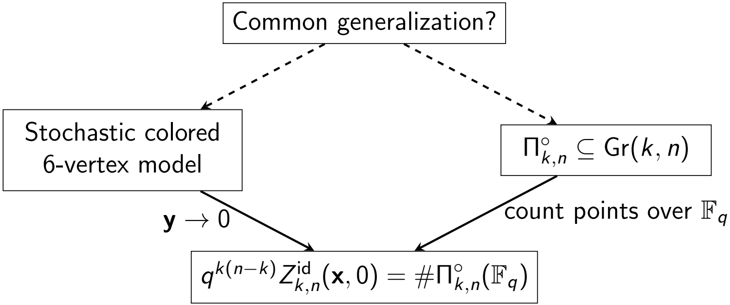
- Flip theorem as $\mathbf{y} \to 0$: $Z^{\mathbb{H},\mathbb{V}}(\mathbf{x}, 0) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x}, 0)$ – lifts to the $\mathrm{Gr}(k, n)$ level

Philosophy



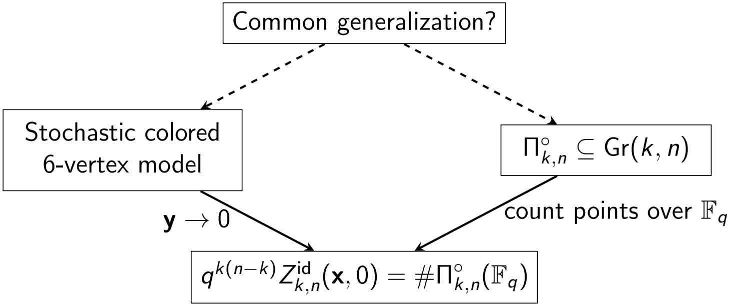
- Flip theorem as $\mathbf{y} \to 0$: $Z^{\mathbb{H},\mathbb{V}}(\mathbf{x}, 0) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x}, 0)$ – lifts to the $\mathrm{Gr}(k,n)$ level
- Proof of flip theorem (Yang–Baxter and Hecke algebra)

Philosophy



- Flip theorem as $\mathbf{y} \to 0$: $Z^{\mathbb{H},\mathbb{V}}(\mathbf{x}, 0) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x}, 0)$ – lifts to the $\mathrm{Gr}(k, n)$ level
- Proof of flip theorem (Yang–Baxter and Hecke algebra) – also lifts to $\mathrm{Gr}(k, n)$:

Philosophy



- Flip theorem as $\mathbf{y} \to 0$: $Z^{\mathbb{H},\mathbb{V}}(\mathbf{x}, 0) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x}, 0)$ – lifts to the $\mathrm{Gr}(k,n)$ level
- Proof of flip theorem (Yang–Baxter and Hecke algebra) – also lifts to $\mathrm{Gr}(k,n)$:

[MS16] Greg Muller and David E. Speyer. Cluster algebras of Grassmannians are locally acyclic. *Proc. Amer. Math. Soc.*, 144(8):3267–3281, 2016.

Same recurrence

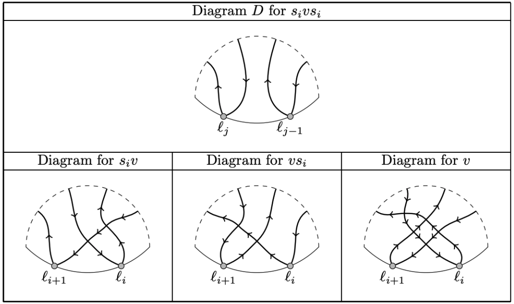


FIGURE 7. Building Postnikov diagrams for s_iv, vs_i and s_ivs_i

[MS16] Greg Muller and David E. Speyer. Cluster algebras of Grassmannians are locally acyclic. *Proc. Amer. Math. Soc.*, 144(8):3267–3281, 2016.

Same recurrence

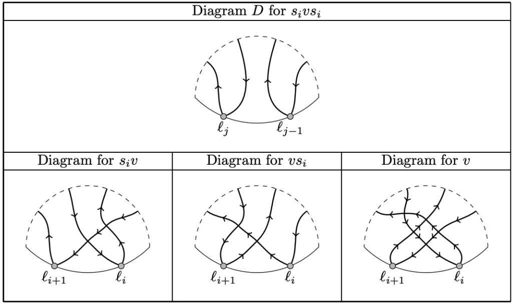


FIGURE 7. Building Postnikov diagrams for s_iv, vs_i and s_ivs_i

[MS16] Greg Muller and David E. Speyer. Cluster algebras of Grassmannians are locally acyclic. *Proc. Amer. Math. Soc.*, 144(8):3267–3281, 2016.

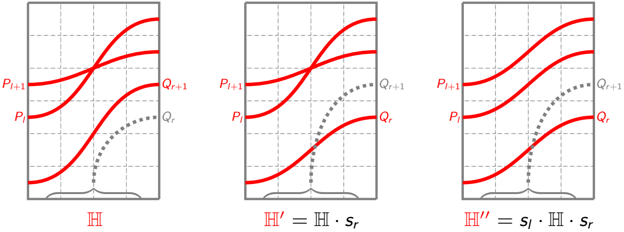


FIGURE 10. The induction step in the proof of Theorem 1.1: one can express $Z[\mathbb{H}, \mathbf{y}]$ recursively in terms of $Z[\mathbb{H}', \mathbf{y}]$ and $Z[\mathbb{H}'', \mathbf{y}]$.

[G., 2020]

Open problems

- Find a common generalization of the 6-vertex model and positroid varieties.

Open problems

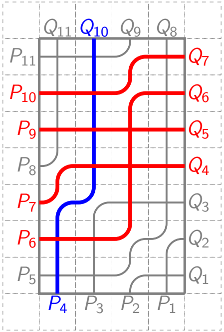
- Find a common generalization of the 6-vertex model and positroid varieties.
- Clarify the relationship with (bumpless, etc) pipe dreams from Schubert calculus.

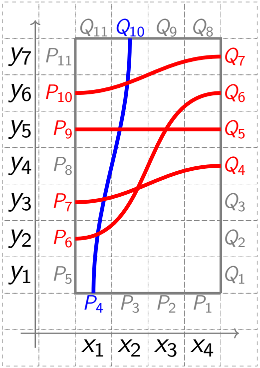
Open problems

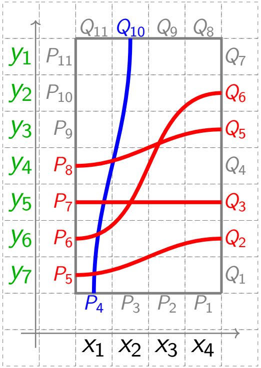
- Find a common generalization of the 6-vertex model and positroid varieties.
- Clarify the relationship with (bumpless, etc) pipe dreams from Schubert calculus.
- How is the flip theorem related to the geometric RSK?

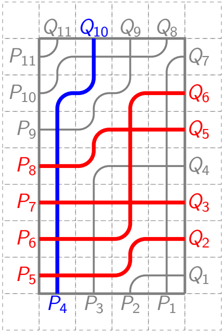
[Dau20] Duncan Dauvergne. Hidden invariance of last passage percolation and directed polymers. arXiv:2002.09459.

Thank you!









$\mathbb{H}^{\pi} = \mathbb{H}$

$\mathbb{V}^{\pi} = \mathbb{V}$

$$Z^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) = Z^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$

$\mathbb{H}^{\pi'} = 180^\circ(\mathbb{H})$

$\mathbb{V}^{\pi'} = \mathbb{V}$