Combinatorics of the Double-Dimer Model

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Dimers in Combinatorics and Cluster Algebras 2020

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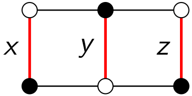
This talk is being recorded

Outline

Kuo condensation

- Today $G = (V_1, V_2, E)$ is a finite bipartite planar graph.
- Let $Z^D(G)$ denote the *partition function*.

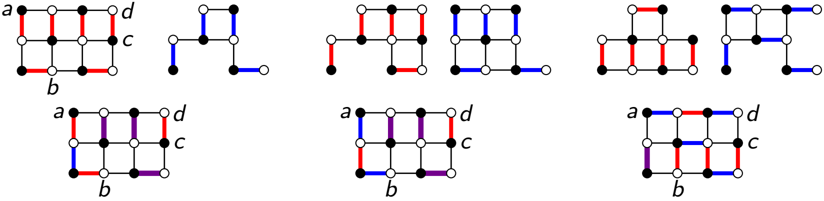
$$Z^D(G) = xyz + x + z$$



Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\}) = Z^D(G-\{a,b\})Z^D(G-\{c,d\}) + Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$



Kuo Condensation

Theorem (Kuo04, Theorem 5.1)

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Examples of non-bijective proofs:

- Fulmek, *Graphical condensation, overlapping Pfaffians and superpositions of Matchings*
- Speyer, *Variations on a theme of Kasteleyn, with Application to the TNN Grassmannian*

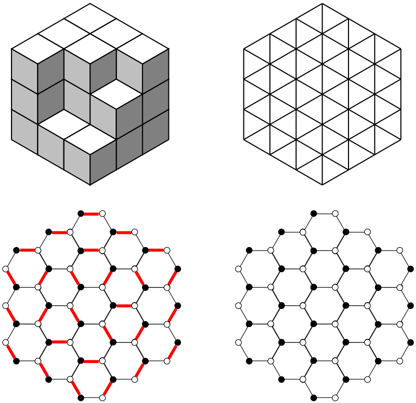
Theorem (Desnanot-Jacobi identity/Dodgson condensation)

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

M_i^j is the matrix M with the ith row and the jth column removed.

Applications of Kuo's work

- Tiling enumeration
  New proof of MacMahon's product formula for the generating function for plane partitions that are subsets of an $r \times s \times t$ box.
- Cluster algebras
  (LM17) Toric cluster variables for the quiver associated to the cone of the del Pezzo surface of degree 6



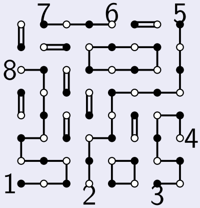
Main result. An analogue of Kuo's theorem for double-dimer configs.

Application: A problem in Donaldson-Thomas theory and Pandharipande-Thomas theory (joint work with Ben Young and Gautam Webb)

Double-dimer configurations

N is a set of special vertices called *nodes* on the outer face of G.

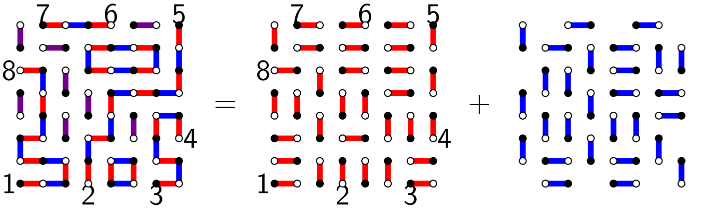
Definition (Double-dimer configuration on $(G, \mathbf{N})$)



Configuration of

- ℓ disjoint loops
- Doubled edges
- Paths connecting nodes in pairs

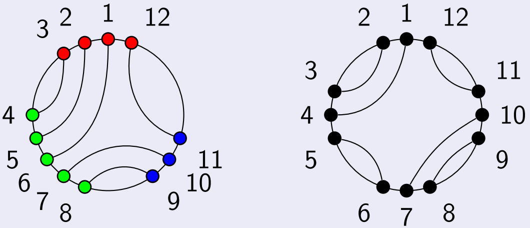
Its weight is the product of its edge weights $\times\ 2^{\ell}$



Tripartite pairings

Definition (Tripartite pairing)

A planar pairing σ of $\mathbf{N}$ is *tripartite* if the nodes can be divided into ≤ 3 sets of circularly consecutive nodes so that no node is paired with a node in the same set.



Tripartite   Not tripartite

We often color the nodes in the sets red, green, and blue, in which case σ has no monochromatic pairs.

Dividing nodes into three sets R, G, and B defines a tripartite pairing.

Main Result

$Z_\sigma^{DD}(G, \mathbf{N})$ denotes the weighted sum of all DD config with pairing σ.

Theorem (J.)

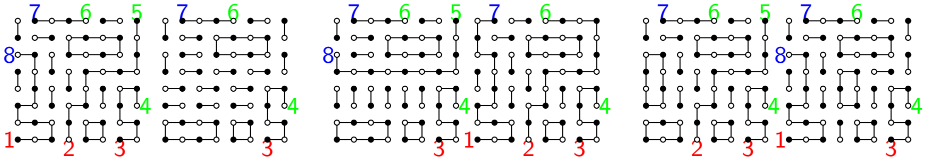
Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathbf{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *If* $\{x, y, w, v\}$ *contains at least one node of each RGB color and* x, y, w, v *appear in cyclic order then*

$$Z_\sigma^{DD}(G, \mathbf{N}) Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\}) =$$

$$Z_{\sigma_{xy}}^{DD}(G, \mathbf{N} - \{x, y\}) Z_{\sigma_{wv}}^{DD}(G, \mathbf{N} - \{w, v\}) + Z_{\sigma_{xv}}^{DD}(G, \mathbf{N} - \{x, v\}) Z_{\sigma_{wy}}^{DD}(G, \mathbf{N} - \{w, y\})$$

Example.

$$Z_\sigma^{DD}(\mathbf{N}) Z_{\sigma_{1258}}^{DD}(\mathbf{N} - 1, 2, 5, 8) = Z_{\sigma_{12}}^{DD}(\mathbf{N} - 1, 2) Z_{\sigma_{58}}^{DD}(\mathbf{N} - 5, 8) + Z_{\sigma_{18}}^{DD}(\mathbf{N} - 1, 8) Z_{\sigma_{25}}^{DD}(\mathbf{N} - 2, 5)$$



Main Result

$Z^{DD}_{\sigma}(G, \mathbf{N})$ denotes the weighted sum of all DD config with pairing σ.

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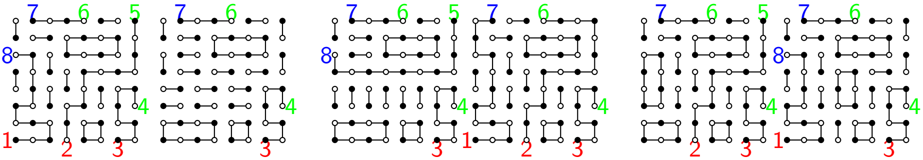
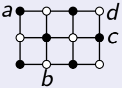


We only need the two nodes of the same RGB color to be opposite in BW color.

Corollaries

Theorem (Kuo04, Theorem 5.1)



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Theorem (J.)

Let $x, y, w, v \in \mathbf{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$*. If* $\{x, y, w, v\}$ *contains at least one node of each RGB color and the two nodes of the same RGB color are opposite in BW color then*

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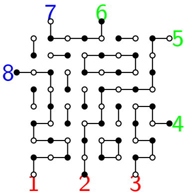

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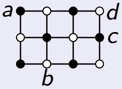

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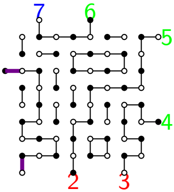
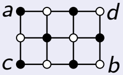


Corollaries

Theorem (Kuo04, Theorem 5.2)



Let vertices a, c, b, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,d\})Z^D(G-\{b,c\})-Z^D(G-\{a,b\})Z^D(G-\{c,d\})$$

Theorem (J.)

Let $x, y, w, v \in \mathbf{N}$ such that $x < w \in V_1$ and $y < v \in V_2$. If $\{x, y, w, v\}$ contains at least one node of each RGB color and the two nodes of the same RGB color are the same in BW color then

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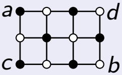

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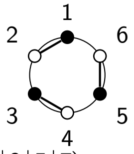
Background: Double-dimer pairing probabilities

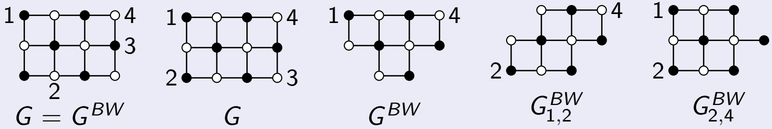


$$\widehat{\Pr}\begin{pmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{pmatrix} = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}$$

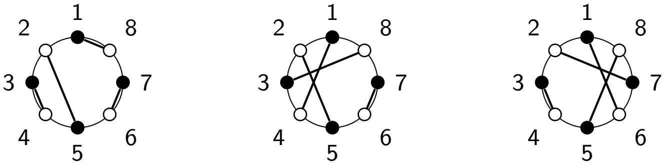
$$\widehat{\Pr}\begin{pmatrix}1 & 3 & 5 & 7\\ 8 & 4 & 2 & 6\end{pmatrix} = X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2} - X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}$$

Definition (KW11a)

$X_{i,j} = \frac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})}$, where $G^{BW} \subseteq G$ only contains nodes that are black and odd or white and even.



- $X_{i,j} = 0$ if i and j have the same parity



$$\widehat{\Pr}\left(\begin{smallmatrix}1 & 3 & 5 & 7\\ 8 & 4 & 2 & 6\end{smallmatrix}\right) = X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2} - X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}$$

- Each term in $\widehat{\Pr}(\sigma)$ is of the form

$$X_\tau := \prod_{(i,j)\in\tau} X_{i,j}, \text{ where } \tau \text{ is an odd-even pairing.}$$

- Kenyon and Wilson made a simplifying assumption that all nodes are black and odd or white and even.

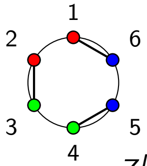
Theorem (KW11a, Theorem 1.3)

$\widehat{Pr}(\sigma)$ is an integer-coeff homogeneous polynomial in the quantities $X_{i,j}$

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\widehat{Pr}(\sigma) = \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]^{i=1,3,\dots,2n-1}_{j=\sigma(1),\sigma(3),\dots,\sigma(2n-1)}.$$



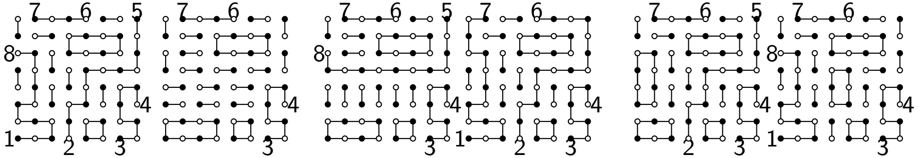
$$\widehat{\Pr}\begin{pmatrix} 1 & 3 & 5 \\ 6 & 2 & 4 \end{pmatrix} = \begin{vmatrix} X_{1,6} & 0 & X_{1,4} \\ X_{3,6} & X_{3,2} & 0 \\ 0 & X_{5,2} & X_{5,4} \end{vmatrix}$$

Since $\widehat{\Pr}(\sigma) := \dfrac{Z^{DD}_{\sigma}(G, \mathbf{N})}{(Z^{D}(G^{BW}))^2}$, the idea of the proof is to combine K-W's matrix with the Desnanot-Jacobi identity:

$$\det(M)\det(M^{i,j}_{i,j}) = \det(M^{i}_{i})\det(M^{j}_{j}) - \det(M^{j}_{i})\det(M^{i}_{j})$$

Example

$$Z^{DD}_{\sigma}(\mathbf{N}) Z^{DD}_{\sigma_{1258}}(\mathbf{N}-1,2,5,8) = Z^{DD}_{\sigma_{12}}(\mathbf{N}-1,2) Z^{DD}_{\sigma_{58}}(\mathbf{N}-5,8) + Z^{DD}_{\sigma_{18}}(\mathbf{N}-1,8) Z^{DD}_{\sigma_{25}}(\mathbf{N}-2,5)$$

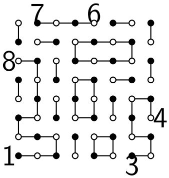


$$M = \begin{pmatrix} X_{1,8} & X_{1,4} & 0 & X_{1,6} \\ X_{3,8} & X_{3,4} & 0 & X_{3,6} \\ X_{5,8} & 0 & X_{5,2} & 0 \\ 0 & X_{7,4} & X_{7,2} & X_{7,6} \end{pmatrix}$$

$$\det(M)\det(M^{1,3}_{1,3}) = \det(M^1_1)\det(M^3_3) - \det(M^3_1)\det(M^1_3)$$

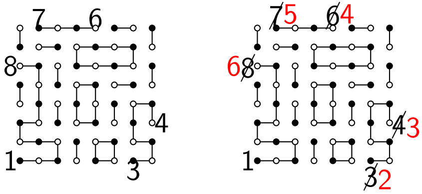
$$\det(M) = \frac{Z^{DD}_{\sigma}(\mathbf{N})}{(Z^D(G^{BW}))^2} \qquad ✔$$

$$\det(M_3^3) \stackrel{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$



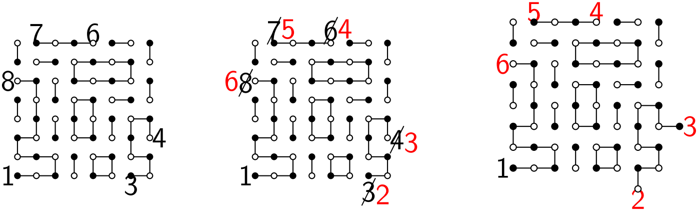
- The nodes are not numbered consecutively.

$$\det(M_3^3) \stackrel{?}{=} \frac{Z_{\sigma_2}^{DD}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$



- Relabel the nodes.
- Node 2 is black and node 3 is white.

$$\det(M_3^3) \overset{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$

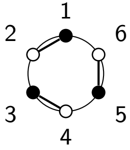


- Add edges of weight 1 to nodes 2 and 3.
- Since $X_{i,j} = \frac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})}$, the K-W matrix for this new graph will have different entries!

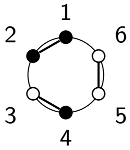
Observation. We need to lift the assumption that the nodes of the graph are black and odd or white and even.

Our Approach

- When the nodes are black and odd or white and even, $G = G^{BW}$, so
$$X_{i,j} = \frac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})} = \frac{Z^D(G_{i,j})}{Z^D(G)}.$$
- Let $Y_{i,j} = \dfrac{Z^D(G_{i,j})}{Z^D(G)}$ and let $\widetilde{\Pr}(\sigma) = \dfrac{Z^{DD}_\sigma(G)}{(Z^D(G))^2}$
- We establish analogues of K-W without their node coloring constraint.

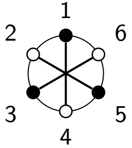


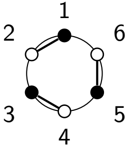
$$\widehat{\Pr}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\middle|\begin{smallmatrix}3\\4\end{smallmatrix}\middle|\begin{smallmatrix}5\\6\end{smallmatrix}\right) = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}$$



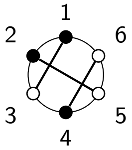
$$\widetilde{\Pr}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\middle|\begin{smallmatrix}3\\4\end{smallmatrix}\middle|\begin{smallmatrix}5\\6\end{smallmatrix}\right) = Y_{1,3}Y_{2,5}Y_{4,6} + Y_{1,5}Y_{2,6}Y_{4,3}$$

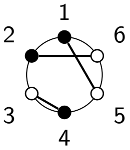
- $X_{i,j} = 0$ if i and j are the same parity
- $Y_{i,j} = 0$ if i and j are the same color





$$\widehat{\Pr}\left(\begin{array}{c|c|c}1&3&5\\2&4&6\end{array}\right) = X_{1,4}X_{2,5}X_{3,6} \;+\; X_{1,2}X_{3,4}X_{5,6}$$





$$\widetilde{\Pr}\left(\begin{array}{c|c|c}1&3&5\\2&4&6\end{array}\right) = Y_{1,3}Y_{2,5}Y_{4,6} \;+\; Y_{1,5}Y_{2,6}Y_{4,3}$$

- Each term in $\widehat{\Pr}(\sigma)$ is of the form

$$X_\tau := \prod_{(i,j)\in\tau} X_{i,j}, \text{ where } \tau \text{ is an odd-even pairing.}$$

- Each term in $\widetilde{\Pr}(\sigma)$ is of the form

$$Y_\rho := \prod_{(i,j)\in\rho} Y_{i,j}, \text{ where } \rho \text{ is an black-white pairing.}$$

A disaster of signs!

Lemma (KW11a, Lemma 3.4)

For odd-even pairings ρ,

$$\mathit{sign}_{OE}(\rho) \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} = (-1)^{\#\ \mathit{crosses\ of}\ \rho}.$$

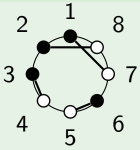
We need a version of this for black-white pairings.

Example ($\text{sign}_{OE}(\rho)$)

If $\rho = \left(\begin{smallmatrix}1 & 3 & 5\\ 6 & 2 & 4\end{smallmatrix}\right)$, then $\text{sign}_{OE}(\rho)$ is the parity of $\left(\frac{6}{2}\ \frac{2}{2}\ \frac{4}{2}\right) = (3\ 1\ 2)$

How to define $\text{sign}(\rho)$ if ρ is black-white?

Example



If $\rho = \left(\begin{smallmatrix}1 & 2 & 3 & 6\\ 7 & 8 & 4 & 5\end{smallmatrix}\right)$, $\text{sign}_{BW}(\rho)$ is the sign of $(3\ 4\ 1\ 2)$.

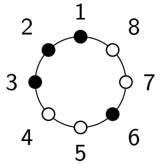
Lemma (KW11a, Lemma 3.4)

For odd-even pairings ρ,

$$sign_{OE}(\rho) \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} = (-1)^{\#\ crosses\ of\ \rho}.$$

Definition

If (i,j) is a pair in a black-white pairing, let $\text{sign}(i,j) = (-1)^{(|i-j|+a_{i,j}-1)/2}$



$a_{7,3} = 1$, so $\text{sign}(7,3) = (-1)^{(|7-3|+1-1)/2} = 1$
$a_{8,3} = 2$, so $\text{sign}(8,3) = (-1)^{(|8-3|+2-1)/2} = -1$

Lemma (J.)

If ρ is a black-white pairing,

$$sign_c(\mathbf{N}) sign_{BW}(\rho) \prod_{(i,j)\in\rho} sign(i,j) = (-1)^{\#\ crosses\ of\ \rho}.$$

Determinant Formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\begin{aligned}\widehat{Pr}(\sigma) &= \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}^{i=1,3,\ldots,2n-1} \\ &= sign_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored diff}} X_{i,j}]_{j=2,4,\ldots,2n}^{i=1,3,\ldots,2n-1}\end{aligned}$$

Theorem (J.)

When σ is a tripartite pairing,

$$\widetilde{Pr}(\sigma) = sign_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored differently}} Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}.$$

More general result

Theorem (J.)

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathbf{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *Then*

$$\begin{aligned} & sign_{OE}(\sigma) sign_{OE}(\sigma'_{xywv}) Z^{DD}_{\sigma}(G, \mathbf{N}) Z^{DD}_{\sigma_{xywv}}(G, \mathbf{N} - \{x, y, w, v\}) \\ = \; & sign_{OE}(\sigma'_{xy}) sign_{OE}(\sigma'_{wv}) Z^{DD}_{\sigma_{xy}}(G, \mathbf{N} - \{x, y\}) Z^{DD}_{\sigma_{wv}}(G, \mathbf{N} - \{w, v\}) \\ & - sign_{OE}(\sigma'_{xv}) sign_{OE}(\sigma'_{wy}) Z^{DD}_{\sigma_{xv}}(G, \mathbf{N} - \{x, v\}) Z^{DD}_{\sigma_{wy}}(G, \mathbf{N} - \{w, y\}) \end{aligned}$$

Corollary

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathbf{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *If* $\{x, y, w, v\}$ *contains at least one node of each RGB color and* x, y, w, v *appear in cyclic order then*

$$Z^{DD}_{\sigma}(G, \mathbf{N}) Z^{DD}_{\sigma_{xywv}}(G, \mathbf{N} - \{x, y, w, v\}) =$$

$$Z^{DD}_{\sigma_{xy}}(G, \mathbf{N} - \{x, y\}) Z^{DD}_{\sigma_{wv}}(G, \mathbf{N} - \{w, v\}) + Z^{DD}_{\sigma_{xv}}(G, \mathbf{N} - \{x, v\}) Z^{DD}_{\sigma_{wy}}(G, \mathbf{N} - \{w, y\})$$

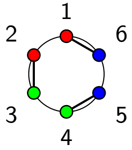
Non-tripartite pairings

The proof of the condensation theorem required taking minors of

$$M = [1_{i,j \text{ RGB-colored differently}}\ Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}$$

Example.

$$\widetilde{\Pr}\begin{pmatrix}1 & 3 & 5\\ 6 & 2 & 4\end{pmatrix} = \begin{vmatrix} 0 & Y_{1,4} & Y_{1,6} \\ Y_{3,2} & 0 & Y_{3,6} \\ Y_{5,2} & Y_{5,4} & 0 \end{vmatrix} = Y_{1,6}Y_{3,2}Y_{5,4} - Y_{1,4}Y_{3,6}Y_{5,2}$$



$$\det(M_{r_1}^{c_4}) = \begin{vmatrix} 0 & Y_{3,6} \\ Y_{5,4} & 0 \end{vmatrix} = \widetilde{\Pr}(35|62)$$

$\det(M_{r_1}^{c_4})$ is equal to the specialization of $\widetilde{\Pr}\begin{pmatrix}1 & 3 & 5\\ 6 & 2 & 4\end{pmatrix}$ obtained by letting $Y_{1,4} = 1$ and $Y_{1,j} = 0$ for all $j \neq 4$.
What happens if we specialize polynomials associated to nontripartite pairings in this way?

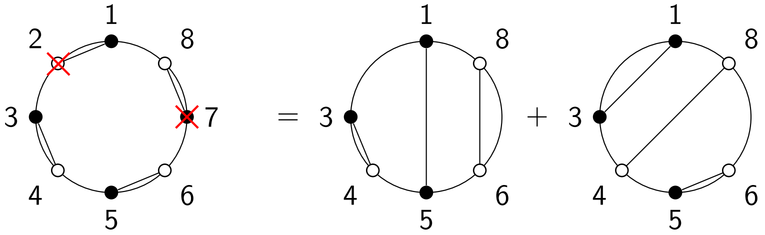
Non-tripartite pairings

Example.

$$\widetilde{\mathrm{Pr}}(12|34|56|78) = Y_{1,2}Y_{3,4}Y_{5,6}Y_{7,8} + Y_{1,2}Y_{3,6}Y_{5,8}Y_{7,4} + Y_{1,4}Y_{3,6}Y_{5,2}Y_{7,8}$$
$$+Y_{1,4}Y_{3,8}Y_{5,6}Y_{7,2} + Y_{1,6}Y_{3,4}Y_{5,8}Y_{7,2} - 2Y_{1,4}Y_{3,6}Y_{5,8}Y_{7,2} + Y_{1,6}Y_{3,8}Y_{5,2}Y_{7,4}$$

Let $Y_{7,2} = 1$ and $Y_{7,j} = 0$ for $j \neq 2$.

$$\begin{aligned}\widetilde{\mathrm{Pr}}_{7,2}(12|34|56|78) &= Y_{1,4}Y_{3,8}Y_{5,6} + Y_{1,6}Y_{3,4}Y_{5,8} - 2Y_{1,4}Y_{3,6}Y_{5,8} \\ &= \widetilde{\mathrm{Pr}}(15|34|68) + \widetilde{\mathrm{Pr}}(13|48|56)\end{aligned}$$



Thank you for listening!

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