The Purity Conjecture for symmetric plabic graphs

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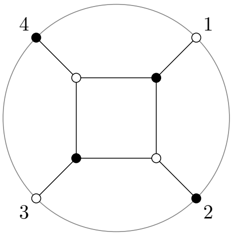
August 3, 2020

Outline

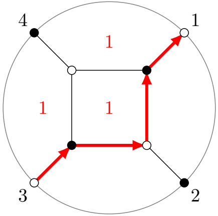
- Plabic graphs and face labels [Postnikov, 2006, Scott, 2006].
- Weakly separated collections, the Purity Conjecture [Leclerc and Zelevinsky, 1998, Scott, 2006, Oh et al., 2015].
- Symmetric plabic graphs, symmetric weakly separated collections.
- The Symmetric Purity Conjecture.
- Idea of the proof.

Plabic graphs



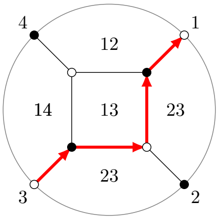
- Class of "planar bicolored" networks, give coordinate charts on Grassmannian [Postnikov, 2006].
  - Related to *total positivity*, cluster structures on Grassmannians [Postnikov, 2006, Scott, 2006, Lam and Galashin, 2019].
  - Weighted paths [Postnikov, 2006], almost perfect matchings [Talaska, 2008, Postnikov et al., 2009, Lam, 2013], face labels [Scott, 2006, Muller and Speyer, 2017].

Trips in plabic graphs



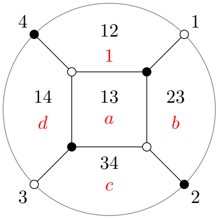
- Turn right at each black vertex, left at each white vertex.
- Label face with index i if it is to the *left* of the trip *ending* at boundary vertex i.

Face labels



- All faces labeled with sets of same size.
- Graph gives coordinate chart on $Gr(k, m)$ where m is number of boundary vertices, k is number of integers in face label.

Plücker coordinates from plabic graphs



- Weights of faces give maximal minors of 2×4 matrix.

$$\begin{bmatrix} 1 & 0 & -b & -\frac{c+bd}{a} \\ 0 & 1 & a & d \end{bmatrix}$$

- $$\Delta_{12} = 1 \qquad \Delta_{23} = b \qquad \Delta_{34} = c \qquad \Delta_{14} = d$$
$$\Delta_{13} = a \qquad \Delta_{24} = \frac{c+bd}{a}$$

Coordinate charts from plabic graphs

- Restricting to positive weights, get isomorphism to *totally positive Grassmannian* $Gr_{>0}(k, m)$ [Postnikov, 2006].
- Minors indexed by plabic graph for $\mathrm{Gr}(k, m)$ form *clusters* in cluster algebra structure on the coordinate ring of $\mathrm{Gr}(k, m)$ [Scott, 2006].
  - These minors form a *total positivity test* for $\mathrm{Gr}(k, n)$.

Weakly separated collections

- Two k-element subsets of $[m]$ are *weakly separated* if when the integers $1, 2, \ldots, m$ are arranged in circle, there is a chord separating $I \backslash J$ from $J \backslash I$.

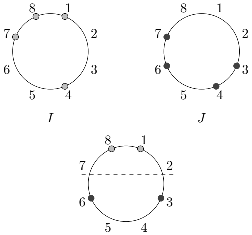


Figure: I, J weakly separated.

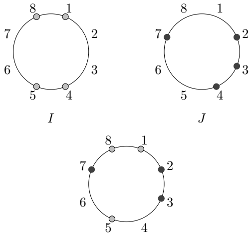


Figure: I, J not weakly separated.

- A collection $\mathcal{C}$ of k-element subsets of $[m]$ is a *weakly separated collection* if I, J are weakly separated for any $I, J \in \mathcal{C}$.

The Purity Conjecture

Conjecture ([Scott, 2006])

A weakly separated collection $\mathcal{C} \subseteq \binom{[m]}{k}$ is maximal by inclusion among all such weakly separated collections if and only if $|\mathcal{C}| = m(m-k)+1$.

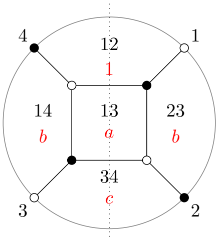
- Related to earlier conjecture of [Leclerc and Zelevinsky, 1998].

Theorem ([Oh et al., 2015])

Let $\mathcal{C} \subseteq \binom{[m]}{k}$ be a weakly separated collection. Then the following are equivalent.

- *$\mathcal{C}$ is maximal by inclusion.*
- *$\mathcal{C}$ is maximal by size, so $|\mathcal{C}| = m(m-k)+1$.*
- *$\mathcal{C}$ is the set of face labels of a plabic graph for $\mathrm{Gr}(k,m)$.*

Symmetric plabic graphs



- Reflection about vertical axis **reverses** colors of vertices.
- Face and its mirror image have same weight.

$$\begin{bmatrix} 1 & 0 & -b & -\frac{c+b^2}{a} \\ 0 & 1 & a & b \end{bmatrix}$$

From symmetric graphs to isotropic subspaces

- Let $\langle e_1, \dots, e_{2n} \rangle$ be the standard basis of $\mathbb{C}^{2n}$. Define bilinear form

$$\langle e_i, e_j \rangle = \begin{cases} (-1)^j & \text{if } i + j = 2n + 1 \\ 0 & \text{otherwise} \end{cases}$$

- Symmetric plabic graphs correspond to matrices whose rows are orthogonal with respect to this form.
- Example:

$$\begin{bmatrix} 1 & 0 & -b & -\frac{c+b^2}{a} \\ 0 & 1 & a & b \end{bmatrix}$$

$$1 \cdot b - 0 \cdot a + (-b) \cdot 1 - \left(-\frac{c + b^2}{a} \right) \cdot 0 = 0$$

Coordinate charts on the Lagrangian Grassmannian.

- The Lagrangian Grassmannian $\Lambda(2n)$ is the subset $\mathrm{Gr}(n, 2n)$ corresponding to maximal isotropic subspaces of with respect to $\langle , \rangle$.

Theorem (K. 2019)

Face labels of symmetric plabic graphs give rational coordinates on $\Lambda(2n)$. Restricting to positive face weights gives an isomorphism to the totally positive part of $\Lambda(2n)$.

Symmetric weakly separated collections

- Fix a positive integer n. For $i \in [2n]$, let $\bar{i} = 2n + 1 - i$.
- For $I \in \binom{[2n]}{n}$, define $\bar{I} = [2n] \backslash \{\bar{i} \mid i \in I\}$.
    - Example: $n = 3$, $I = \{2, 4, 5\}$
    - $\bar{I} = [6] \backslash \{5, 3, 2\} = \{1, 4, 6\}$
- A weakly separated collection $\mathcal{C}$ is *symmetric* if $\bar{I} \in \mathcal{C}$ for each $I \in \mathcal{C}$.

Main Theorem

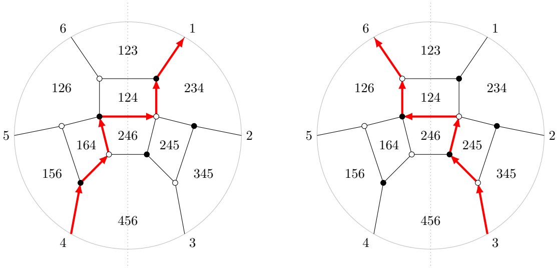
Theorem (K., 2019)

Let $\mathcal{C}$ be a **symmetric** *weakly separated collection in $\binom{[2n]}{n}$. Then the following are equivalent.*

1. *$|\mathcal{C}|$ is maximal by size, so $|\mathcal{C}| = 2n^2 + 1$.*
2. *$\mathcal{C}$ is maximal by inclusion among all such symmetric weakly separated collections.*
3. *$\mathcal{C}$ is the set of face labels of a symmetric plabic graph for $\Lambda(2n)$.*

- The equivalence $(1) \Leftrightarrow (2)$ has been generalized to a larger family of collections by [Danilov et al., 2019].

Trips in symmetric plabic graphs



- Notice: face is to the left of trip ending at i iff reflection is to the **right** of trip ending at i.
    - Label of face contains i iff label of reflection **does not** contain $\bar{i}$.
- Face has label I, reflection has label $\bar{I}$.

Reflecting face labels

- I has a *full pair* $\{i, \bar{i}\}$ if $i, \bar{i} \in I$. Empty pairs are defined similarly.

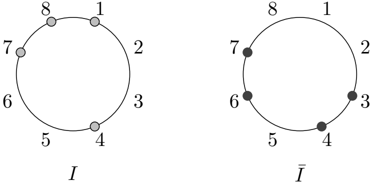


Figure: I has full pair at $\{1, 8\}$, empty pair at $\{3, 6\}$.

- Get $\bar{I}$ from I by exchanging "full pairs" and "empty pairs."

Admissible elements

- I is *admissible* if I and $\bar{I}$ are weakly separated.
- Every face label of a symmetric plabic graph is admissible.

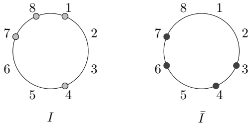


Figure: $I, \bar{I}$ weakly separated.

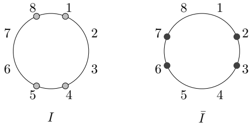


Figure: $I, \bar{I}$ not weakly separated.

- I is admissible if and only if I does not have a full pair between two empty pairs, or an empty pair between two full pairs.

Left, right and center elements

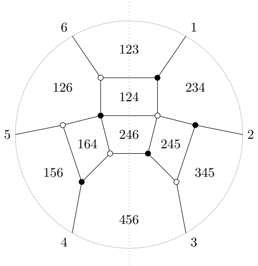


Figure: A symmetric plabic graph.

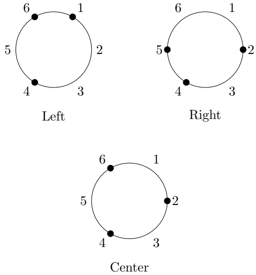


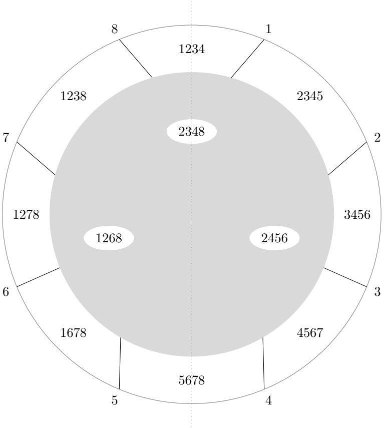
Figure: Some face labels.

- Labels on the left have full pairs above empty pairs.
- Labels on the right have empty pairs above full pairs.
- Labels in the center have no full or empty pairs.

Total positivity tests for $\Lambda(2n)$

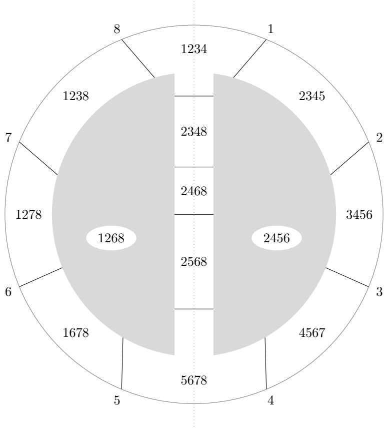
- Take maximal symmetric weakly separated collection, look at *center* and *left* elements only.
- These give rational coordinates on $\Lambda(2n)$.
- Total positivity test on $\Lambda(2n)$, minimal by inclusion.
- Question: are these cluster variables in a cluster structure?

Proof sketch



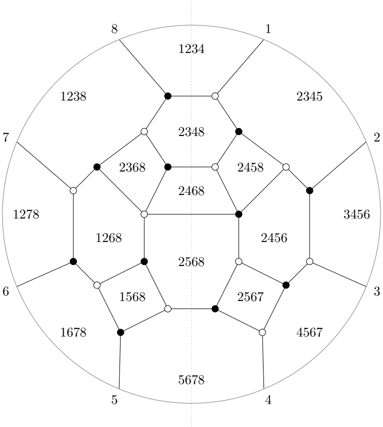
- Start with symmetric weakly separated collection *not* maximal by size.
- Build symmetric plabic graph which has all elements as face labels.

Fill in the center



- Can always add face labels to build a "chain" along midline.
- Example: start with 1234, successively replace elements to get 5678.

Fill in the rest



- Once center "filled in", any J weakly separated from current set of face labels of guaranteed to be admissible.
- Add $\{J, \bar{J}\}$, repeat until collection maximal by size.

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