Categorification of perfect matchings

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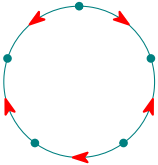
Toy model: perfect matchings on a circle

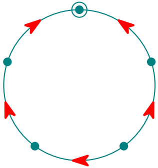


On a circular graph $(\mathcal{C}_0, \mathcal{C}_1)$ with n vertices,
a *(perfect) matching* is a choice of orientation for each edge.

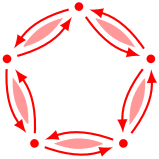
A matching is specified by its label $J = (J_\bullet, J_\circ)$ with $J_\bullet \subseteq \mathcal{C}_1$ being the anti-clockwise edges and $J_\circ$ the clockwise edges.

A matching has *chirality* (helicity) $k = (k_\bullet, k_\circ) \in \mathbb{N}^{\{\bullet,\circ\}}$,
where $k_\bullet = \#J_\bullet$ and $k_\circ = \#J_\circ$, so that $k_\bullet + k_\circ = n$.

A new matching of the same chirality is obtained by flipping a source to a sink or vice versa.
Chirality is the only invariant of flipping.



Cochains on a closed string



Fatten the circle to a quiver with faces $Q = (Q_0, Q_1, Q_2)$, i.e. a 2-complex s.t. ∂f is an oriented cycle, for all $f \in Q_2$.

For any such Q, a *matching* is a function $\mu \in \mathbb{N}^{Q_1}$ s.t $d\mu = 1$, on all faces $f \in Q_2$, in ptic, in lattice $\mathbb{M} = \{\mu \in \mathbb{Z}^{Q_1} : d\mu \in c(\mathbb{Z})\}$.

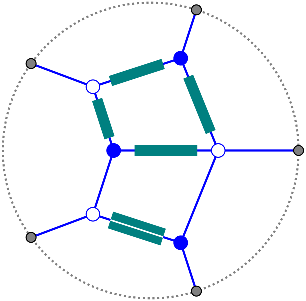
$$
\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{c} & \mathbb{Z}^{Q_0} & \xrightarrow{d} & \mathbb{Z}^{Q_1} & \xrightarrow{d} & \mathbb{Z}^{Q_2} \\
\| & & \| & & \uparrow i & & \uparrow c \\
\mathbb{Z} & \xrightarrow{c} & \mathbb{Z}^{Q_0} & \xrightarrow{d} & \mathbb{M} & \xrightarrow{\deg} & \mathbb{Z}
\end{array}
$$

d is coboundary, c is constants, i is inclusion, deg is restriction of d. Now flip is adding/subtracting $d(s_i)$ for s_i basic in $\mathbb{Z}^{Q_0}$.

Denote by $\mathbb{M}^+ = \mathbb{M} \cap \mathbb{N}^{Q_1}$ the cone of *multi-matchings*.

For string, rank $\mathbb{M} = n + 1$ and $\mathbb{M}^+$ is the cone on a unit n-cube.

Another example: double (or r-fold) dimers



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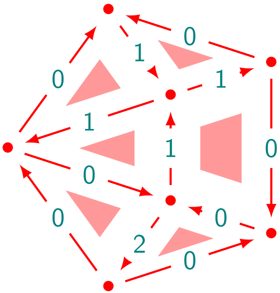


$\deg \mu = 2$

Question: why is $\mathrm{wt}(\Sigma) = 2$? *Guess:* because $\chi(\mathbb{P}^1) = 2$ and some appropriate quiver Grassmannian is $\mathbb{P}^1$.

Chirality revisited

Let $\mathbb{H}^1 = \mathbb{M}/d(\mathbb{Z}^{Q_0})$ and $h\colon \mathbb{M} \to \mathbb{H}^1$ be the quotient.

Note: $\deg = \mathrm{dg} \circ h$ for $\mathrm{dg}\colon \mathbb{H}^1 \to \mathbb{Z}$ and $\mathrm{dg}^{-1}(0) = H^1(Q)$.

For the closed string: $\mathbb{H}^1 \cong \{(h_\bullet, h_\circ) \in \mathbb{Z}^{\{\bullet,\circ\}} : h_\bullet + h_\circ \in n\mathbb{Z}\}$.

Explicitly, write $Q_1 = Q_1^\bullet \cup Q_1^\circ$ and, for $* \in \{\bullet, \circ\}$, define two closed 1-cycles $a_* = \sum_{a \in Q_1^*} a$. Then $h_*(\mu) = \mu(a_*) = \sum_{a \in Q_1^*} \mu(a)$ and $\deg(\mu) = \frac{1}{n}(h_\bullet(\mu) + h_\circ(\mu))$.

Fixed chirality $k = (k_\bullet, k_\circ)$ and define $\mathbb{M}_k = h^{-1}\langle k \rangle$, then $\operatorname{rank} \mathbb{M}_k = n$ and $\mathbb{Z} \xrightarrow{c} \mathbb{Z}^{Q_0} \xrightarrow{d} \mathbb{M}_k \xrightarrow{\deg} \mathbb{Z}$ is exact.

Fact: $\mathbb{M}_k \cong$ the sublattices of the weight lattice of $GL(n)$ that grade the homogeneous coordinate rings $\mathbb{C}[\hat{\mathrm{Gr}}^n_{k_\bullet}]$ and $\mathbb{C}[\hat{\mathrm{Gr}}^n_{k_\circ}]$ and deg gives the usual degree (Plücker coords Δ_J have deg 1).

Categorification of matchings

Let $Z = \mathbb{C}[[t]]$ and $Q = (Q_0, Q_1, Q_2)$ be a quiver with faces s.t.

(i) the associated topological space $|Q|$ is connected

(ii) every arrow $a \in Q_1$ is in the boundary of some face $f \in Q_2$

(iii) Q admits virtual matchings, i.e. $\deg \mathbb{M} \to \mathbb{Z}$ is surjective.

The category of *matrix factorizations* $\mathrm{MF}(Q; t)$ consists of representations M, ϕ of Q s.t.

(i) each $M_i : i \in Q_0$ is a f.g. free Z-module of rank $r := \mathrm{rk}\, M$

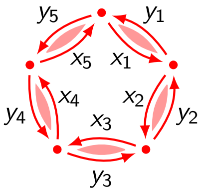
(ii) the maps $\phi_a \colon M_{ta} \to M_{ha}$ satisfy $\phi_{a_s} \circ \cdots \circ \phi_{a_1} = t$,
when $a_s + \cdots + a_1 = \partial f$ is the boundary of a face $f \in Q_2$.

There is a natural invariant $\nu \colon \mathrm{K}(\mathrm{MF}(Q; t)) \to \mathbb{M} \colon [M] \mapsto \nu_M$ given by $\nu_M(a) = \dim \mathrm{coker}\, \phi_a$ and such that $\deg \nu_M = \mathrm{rk}\, M$.

For each (deg 1) matching $\mu \in \mathbb{M}^+$, there is a rk 1 rep'n $M(\mu), \phi$ with $\nu_{M(\mu)} = \mu$, given by $M(\mu)_i = Z$ and $\phi_a = t^{\mu(a)}$.

A flip corresponds to simple extension/shortening of the rep'n.

The closed string categorified



For $x_j \in Q_1^\circ$, $y_j \in Q_1^\bullet$ and fixed $k = (k_\bullet, k_\circ)$, define Z-algebra C_k as path algebra ZQ mod $xy = t = yx$, $x^{k_\bullet} = y^{k_\circ}$ ($\Rightarrow$ $x^n = t^{k_\circ}$, $y^n = t^{k_\bullet}$).

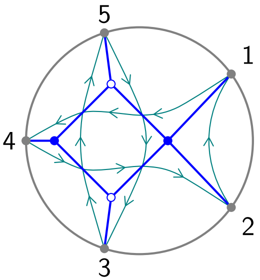
The category CM C_k of f.g. C_k-modules free over Z is a full exact subcategory of $\mathrm{MF}(Q; t)$ and $\nu\colon \mathrm{K}(\mathrm{CM}\, C_k) \xrightarrow{\cong} \mathbb{M}_k \subseteq \mathrm{Wt}\, GL(n)$.

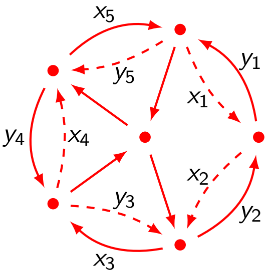
CM C_k contains $M(\mu)$ for all μ of chirality k; these are all the rk 1 modules (up to isom).

Theorem [JKS1] There is a cluster character $\Psi\colon$ CM $C_k \to \mathbb{C}[\hat{\mathrm{Gr}}_k^n]$ such that wt $\Psi_M = \nu_M$. In particular, $\Psi_{M(J)} = \Delta_J$.

Fact: C_k is *thin*, i.e. each component $e_i C e_j$, for $i, j \in Q_0$, is a free Z-module of rank 1. Hence, for all $i \in Q_0$, projectives $\mathcal{P}_i = Ce_i$ and (CM-)injectives $\mathcal{I}_i = (e_i C)^\vee := \mathrm{Hom}_Z(e_i C, Z)$ are matching modules $M(J)$, in fact, for J some cyclic interval.

Plabic graph G and dual quiver with faces Q





Arrow directions follows strands. Boundary arrows and backwards paths in boundary faces are $x = x^\circ$ or $y = x^\bullet$ from orientation.

Dimer algebra A has $\mathrm{CM}\,A = \mathrm{MF}(Q; t)$, with $\mathrm{K}(\mathrm{CM}\,A) = \mathbb{M}$ and all rk 1 modules are matching modules $M(\mu)$.

All $M \in \mathrm{CM}\,A$ satisfy chirality relation $x^{k_\bullet} = y^{k_\circ}$ for some fixed k.

Matchings on G are cochains on Q (Poincaré duality)

$$\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{\mathsf{c}} & \mathbb{Z}^{Q_0} & \xrightarrow{\mathsf{d}} & \mathbb{Z}^{Q_1} & \xrightarrow{\mathsf{d}} & \mathbb{Z}^{Q_2} \\
\| & & \| & & \uparrow \mathsf{i} & & \uparrow \mathsf{c} \\
\mathbb{Z} & \xrightarrow{\mathsf{c}} & \mathbb{Z}^{Q_0} & \xrightarrow{\mathsf{d}} & \mathbb{M} & \xrightarrow{\deg} & \mathbb{Z}
\end{array}$$

Since $|Q|$ is a disc, both horizontal sequences are exact and hence rank $\mathbb{M} = \#Q_0$.

There is a *bdry value map* $\mathfrak{d}\colon \mathbb{M} \to \mathbb{M}_k$ (compatible with deg) that is dual to inclusion of chains: path $x^* \mapsto \sum$ arrows in x^*.

Explicitly $\mathfrak{d}\mu = J = (J_\bullet, J_\circ)$, where $J_* = \{j \in \mathcal{C}_1 : \mu(x_j^*) = 1\}$.

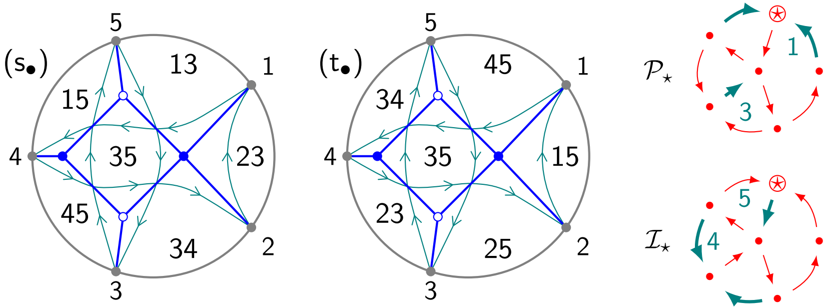
The restriction $\rho_{AC}\colon$ CM $A \to$ CM C categorifies $\mathfrak{d}$.

Projectives and injectives

Consistency $\Rightarrow A$ is thin, so projectives $\mathcal{P}_i = Ae_i$ and injectives $\mathcal{I}_i = (e_i A)^\vee$ are matching modules $M(\mu)$ .. but which?

[Mu-Sp] define bases of matchings $\mathfrak{m}^{\mathsf{s}/\mathsf{t}} \colon \mathbb{Z}^{Q_0} \to \mathbb{M}$, whose bdry values $\partial\mathfrak{m}_j^{\mathsf{s}/\mathsf{t}}$ give source (s) and target (t) labellings for G.

Prop [CKP] For all $j \in Q_0$, we have $[\mathcal{P}_j] = \mathfrak{m}_j^{\mathsf{s}}$ and $[\mathcal{I}_j] = \mathfrak{m}_j^{\mathsf{t}}$.



Boundary algebra, necklace and positroid

Bdry algebra $B = eAe$, where $e = \sum_{i \in \mathcal{C}_0} e_i$ is bdry idempotent.

Restriction ρ_{AC} factorises as CM $A \xrightarrow{\rho_{AB}}$ CM $B \xrightarrow{\rho_{BC}}$ CM C, where $\rho_{AB} \colon X \mapsto eX$ and ρ_{BC} is a fully faithful embedding.

If $i \in \mathcal{C}_0$, then $\rho_{AB} \colon Ae_i \mapsto Be_i$ and $(e_i A)^\vee \mapsto (e_i B)^\vee$, so these are the matching modules $M(N_i)$ and $M(N'_i)$ for necklace N and reverse necklace N'.

In other words, the necklace is B.

Matching module $M(J)$ is in CM B iff J is in the positroid.

Projective resolution

Can view $\mathfrak{m} = \mathfrak{m}^s$ as the map $\mathsf{K}(\mathcal{P}A) \xrightarrow{\cong} \mathsf{K}(\mathsf{CM}\, A)$ induced by inclusion of category $\mathcal{P}A$ of projective A-modules, thus $\mathfrak{m}^{-1}$ comes from projective resolution.

Thm [CKP] Each $M = M(\mu)$ in $\mathsf{CM}\, A$ has a projective resolution

$$\bigoplus_{\substack{a\in\mu \\ int}} Ae_{ta} \to \bigoplus_{a\notin\mu} Ae_{ha} \to \bigoplus_{i\in Q_0} Ae_i$$

[Ma-Sc] define weights for internal arrows $\mathtt{wt}(a) \in \mathbb{Z}^{Q_0} = \mathsf{K}(\mathcal{P}A)$.

Cor [CKP] For $\mu \in \mathbb{M}$,

$$\mathfrak{m}^{-1}(\mu) = \sum_{\substack{a\in Q_1 \\ ext}} \mu(a)[P_{ha}] + \deg(\mu) \overbrace{\sum_{\substack{i\in Q_0 \\ int}} [P_i]}^{\mathtt{wt}(G)} - \overbrace{\sum_{\substack{a\in Q_1 \\ int}} \mu(a)\mathtt{wt}(a)}^{\mathtt{wt}(\mu)}$$

Prop [CKP] For all $j \in Q_0$, we have $\mathfrak{m}^{-1}(\mathfrak{m}^s_j) = [P_j]$.

Newton-Okounkov cone

The restriction functor $\rho_{AB}\colon \mathrm{CM}\,A \to \mathrm{CM}\,B\colon X \mapsto eA \otimes_A X$ has a right adjoint $F\colon \mathrm{CM}\,B \to \mathrm{CM}\,A\colon M \mapsto \mathrm{Hom}_B(eA, M)$.

Here the counit $\eta_X\colon X \to FeX$ is an embedding, i.e., if $eX = M$, then $X \subseteq FM$, so FM is the maximal module which restricts to M.

For $M(J)$ in $\mathrm{CM}\,B$, $FM(J)$ is a matching module $M(\mu)$ and μ is the minimal matching with $\partial\mu = J$ in the flip partial order.

Claim [JKS3] For $M \in \mathrm{CM}\,C$, i.e. the $\hat{\mathrm{Gr}}_k^n$ case, $z^{[FM]}$ is the leading monomial (a la [Ri-Wi]) in network coords of the clus. char. Ψ_M.

See [JKS2] for $\Psi_{M(J)} = \Delta_J$, which is given in network coords by the dimer partition function:

$$\mathcal{Z}_J = \sum_{\mu:\partial\mu=J} z^{\mu}$$

Expectation: (a) The set $\{[FM] \in \mathbb{M} : \mathrm{M} \text{ in } \mathrm{CM}\,C\}$ is precisely the integral points in the Ri-Wi Newton-Okounkov cone for $\hat{\mathrm{Gr}}_k^n$.
(b) a basis of $\mathbb{C}[\hat{\mathrm{Gr}}_k^n]$ is given by $\{\Psi_M : M \text{ general in } \mathrm{CM}\,C\}$.
(c) Similar holds for positroid $\hat{\mathrm{Gr}}_\pi$, by replacing C by B.

Background: network torus and Muller-Speyer twist

$\mathbb{M} \supseteq deg^{-1}(0) \cong \mathbb{Z}^{Q_0}/c\mathbb{Z}$, which is the character lattice of the usual network torus, in monodromy coordinates.

Thus $\mathbb{M}$ is the character lattice of a torus $\mathbb{M}^*$ that lifts the network torus to the positroid cone $\hat{\mathrm{Gr}}^{\circ}_{\pi}$, using the dimer part. fun.

$$\mathbb{C}\left[\hat{\mathrm{Gr}}^{\circ}_{\pi}\right] \to \mathbb{C}\left[\mathbb{M}^*\right] : \Delta_J \mapsto \mathcal{Z}_J := \sum_{\mu : \partial\mu = J} z^{\mu}$$

Note: for $J \in N$, i.e. Δ_J frozen, $\mathcal{Z}_J$ is a monomial so invertible.

Thm: [Mu-Sp] There is an automorphism $\tau \colon \hat{\mathrm{Gr}}^{\circ}_{\pi} \to \hat{\mathrm{Gr}}^{\circ}_{\pi}$ s.t.

$$\begin{array}{ccc}
\mathbb{C}\left[\mathbb{M}^*\right] & \xrightarrow{(-\mathfrak{m}^{-1})^{*\cdot}} & \mathbb{C}\left[(\mathbb{C}^*)^{Q_0}\right] \\
\text{network} \uparrow & & \uparrow \text{cluster} \\
\mathbb{C}\left[\hat{\mathrm{Gr}}^{\circ}_{\pi}\right] & \xrightarrow{\tau^{\cdot}} & \mathbb{C}\left[\hat{\mathrm{Gr}}^{\circ}_{\pi}\right]
\end{array}$$

Application: Marsh-Scott twist

Rearrange the $\mathfrak{m}^{-1}$ formula, when μ is a (deg 1) matching to get

$$\mathtt{wt}(\mu) - \mathtt{wt}(G) = \sum_{a \in \partial\mu} [P_{ha}] - \mathfrak{m}^{-1}(\mu) \qquad (*)$$

Recall: [Ma-Sc] define a twist $\sigma_\bullet \colon \hat{\mathrm{Gr}}_k^n \dashrightarrow \hat{\mathrm{Gr}}_k^n$ and prove that

$$\sigma_\bullet^\cdot(\Delta_J) = \mathcal{Z}_J^{MS} := z^{-\mathtt{wt}(G)} \sum_{\mu : \partial\mu = J} z^{\mathtt{wt}(\mu)} \quad \text{in cluster coords}$$

Define $\mathfrak{p}^\bullet \colon \mathbb{M}_k \to \mathbb{Z}^{Q_0} \colon J \mapsto \sum_{a \in J_\bullet} [P_{ha}]$.

Then $\partial\mathfrak{m}\mathfrak{p}^\bullet([M]) = [P^\bullet M] \in \mathbb{M}_k$ for a projective cover $P^\bullet M \to M$.

$$(*) \Rightarrow \qquad \mathcal{Z}_J^{MS} = z^{\mathfrak{p}^\bullet(J)} \sum_{\mu : \partial\mu = J} z^{-\mathfrak{m}^{-1}(\mu)}$$

Thm [CKP] For $M(J)$ in CM C, we have $\sigma_\bullet^\cdot(\Delta_J) = \Psi_{\Omega^\bullet M(J)}$, where $\Omega^\bullet M$ is the syzygy ker $P^\bullet M \to M$.

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