Tiling Shuffling Phenomenon

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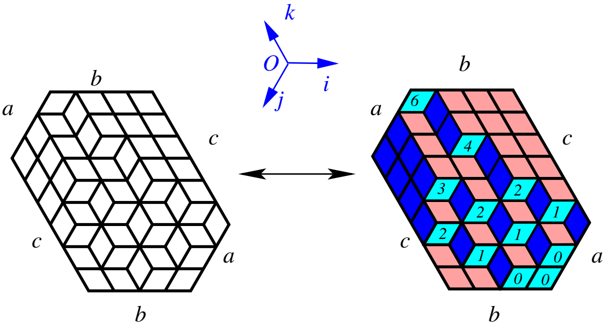
MacMahon's Theorem

Theorem (MacMahon)

$$\mathrm{PP}_q(a,b,c) := \sum_{\pi} q^{vol(\pi)} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{t=1}^{c} \frac{1-q^{i+j+t-1}}{1-q^{i+j+t-2}},$$

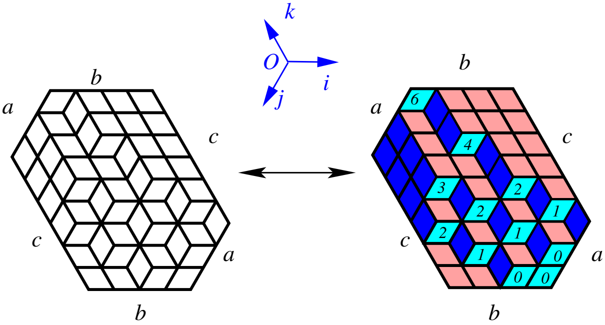
where the sum is taken over all plane partitions π *fitting in an* $a\times b\times c$ *box.*



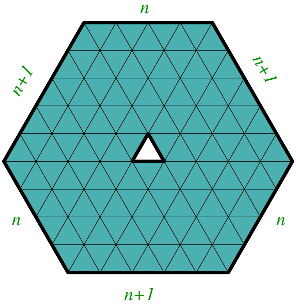
Theorem (MacMahon 1900)

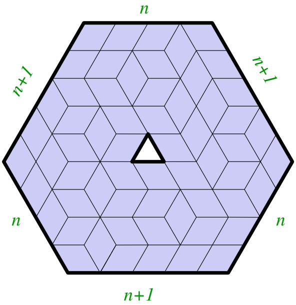
The number of (lozenge) tilings of a centrally symmetric hexagon $Hex(a,b,c)$ *of sides* a,b,c,a,b,c *on the triangular lattice is*

$$\mathrm{PP}(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{t=1}^{c}\frac{i+j+t-1}{i+j+t-2}$$



Punctured Hexagon: James Propp's Problem



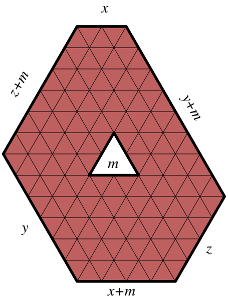


Open Problem (Propp 1997)

Find an explicit formula for the number of tilings of a hexagon of sides $n, n+1, n, n+1, n, n+1$ with the central unit triangle removed.

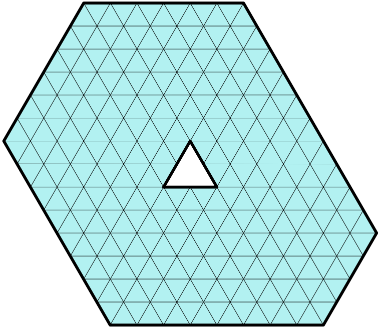
This is Problem 2 on his list of 20 open problems in the field of enumeration of tilings.

Ciucu–Eisenkölbl–Krattenthaler–Zare's cored hexagon



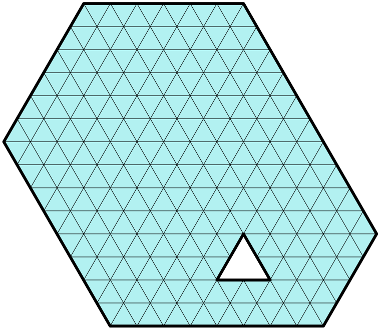
- Ciucu–Eisenkölbl–Krattenthaler–Zare (2001) generalized the above results by extending the size of the hole.
- Unit triangle is replaced by a triangle of an abritrary side.
- The triangular hole is at the 'center' of the hexagon of sides $a, b+m, c, a+m, c+m, b$.

Example



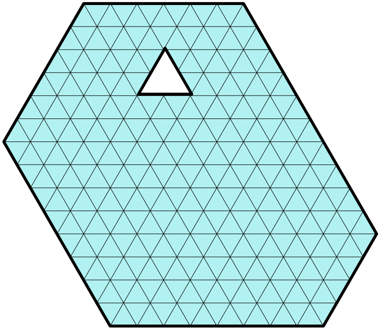
$2^6 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 13^3 \cdot 17^3 \cdot 19$

Example



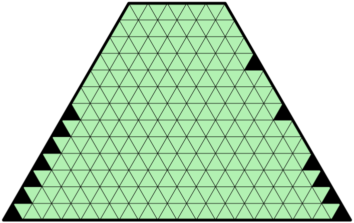
$2^5 \cdot 7 \cdot 11^2 \cdot 13^3 \cdot 17^3 \cdot 19 \cdot 71$

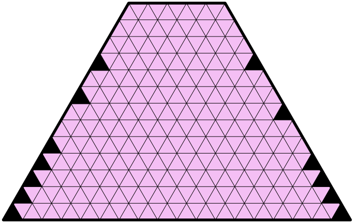
Example



$2^6 \cdot 11 \cdot 13^3 \cdot 17^3 \cdot 19 \cdot 281$

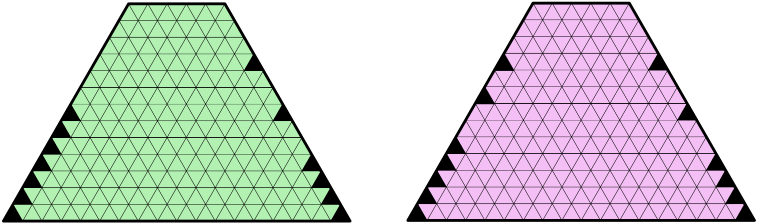
Example





- The left tiling number: $2^5 \cdot 3 \cdot 7^3 \cdot 11^3 \cdot 13^4 \cdot 17$

Example



- The left tiling number: $2^5 \cdot 3 \cdot 7^3 \cdot 11^3 \cdot 13^4 \cdot 17$
- The right tiling number: $2^6 \cdot 7^3 \cdot 11 \cdot 13^4 \cdot 17 \cdot 2683$

Shuffling Phenomenon

- The tiling number of punctured regions are not given by simple product formulas.

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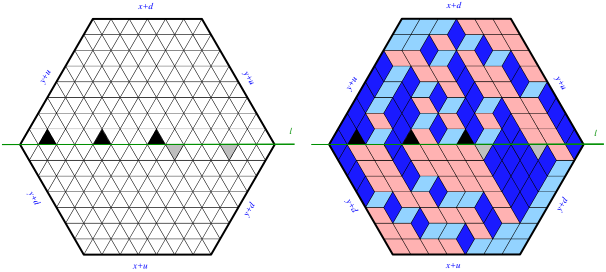
- A small modification (in the position, orientation, side-length, etc.) of the region would lead to unpredictable change in the tiling number.
- However, in certain cases, our modifications change the tiling number by only a simple multiplicative factor.

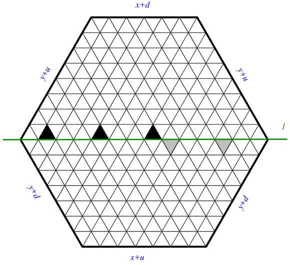
First Example: Doubly-dented hexagon



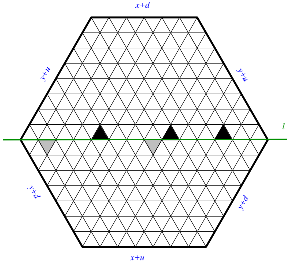
(a) (b)

- Position set of upper holes $U = \{s_1, s_2, \ldots, s_u\} \subset [x + y + u + d]$
- Position set of lower holes $D = \{t_1, t_2, \ldots, t_d\} \subset [x + y + u + d]$
- Assume $U \cap D = \emptyset$.
- Doubly-dented hexagon: $H_{x,y}(U, D)$

Shuffling the holes



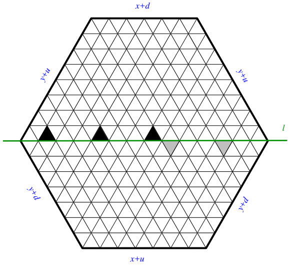
(a)



(b)

- $U = \{s_1, s_2, \ldots, s_u\} \to U' = \{s'_1, s'_2, \ldots, s'_u\}$
- $D = \{t_1, t_2, \ldots, t_d\} \to D' = \{t'_1, t'_2, \ldots, t'_d\}$
- $H_{x,y}(U, D) \to H_{x,y}(U', D')$

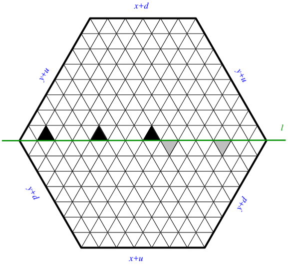
Shuffling the holes



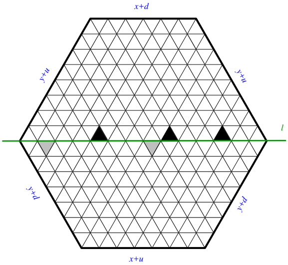
(a) (b)

- Tiling number of $H_{x,y}(U, D)$: $2^9 \cdot 3^5 \cdot 5^3 \cdot 7^4 \cdot 20107$

Shuffling the holes



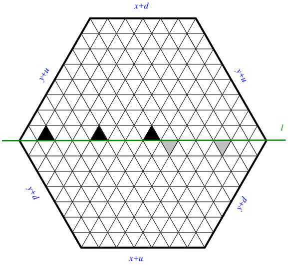
(a)



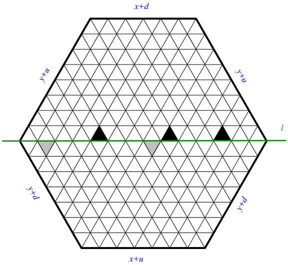
(b)

- Tiling number of $H_{x,y}(U, D)$: $2^9 \cdot 3^5 \cdot 5^3 \cdot 7^4 \cdot 20107$
- Tiling number of $H_{x,y}(U', D')$: $2^{11} \cdot 3^3 \cdot 5^3 \cdot 7^5 \cdot 20107$

Shuffling the holes



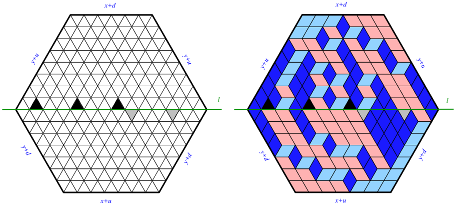
(a)



(b)

- Tiling number of $H_{x,y}(U, D)$: $2^9 \cdot 3^5 \cdot 5^3 \cdot 7^4 \cdot 20107$
- Tiling number of $H_{x,y}(U', D')$: $2^{11} \cdot 3^3 \cdot 5^3 \cdot 7^5 \cdot 20107$
- The ratio of tilings: $2^{-2} \cdot 3^2 \cdot 7^{-1}$

Shuffling Theorem



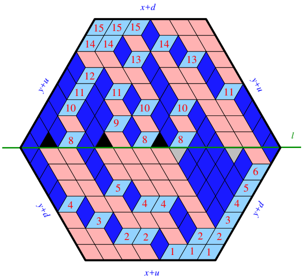
(a) (b)

Theorem (Shuffling Theorem)

For $U = \{s_1, s_2, \dots, s_u\}$, $D = \{t_1, t_2, \dots, t_d\}$, $U' = \{s'_1, s'_2, \dots, s'_u\}$, $D' = \{t'_1, t'_2, \dots, t'_d\}$ *of* $[x + y + n]$, *such that* $U \cup D = U' \cup D'$ *and* $U \cap D = U' \cap D' = \emptyset$

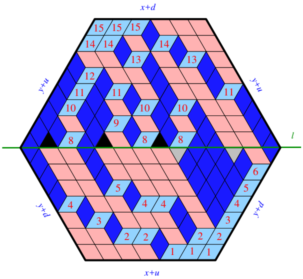
$$\frac{\mathrm{M}(H_{x,y}(U, D))}{\mathrm{M}(H_{x,y}(U', D'))} = \prod_{1 \le i < j \le u} \frac{s_j - s_i}{s'_j - s'_i} \cdot \prod_{1 \le i < j \le d} \frac{t_j - t_i}{t'_j - t'_i} \tag{1}$$

- A right lozenge is weighted by q^t, where t is the distance to the base of the hexagon.



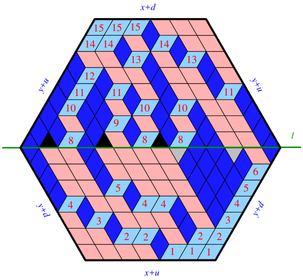
q-Shuffling Theorem

- A right lozenge is weighted by q^t, where t is the distance to the base of the hexagon.
- A tiling is weighted by the product of weights of all lozenges



q-Shuffling Theorem

- A right lozenge is weighted by q^t, where t is the distance to the base of the hexagon.
- A tiling is weighted by the product of weights of all lozenges
- $\mathrm{M}_q(R)$ is the sum of weights of all tilings of R



Theorem (L. –Rohatgi 2019)

$$\frac{\mathrm{M}_q(H_{x,y}(U,D))}{\mathrm{M}_q(H_{x,y}(U',D'))} = q^C \cdot \prod_{1\leq i<j\leq u} \frac{q^{s_j} - q^{s_i}}{q^{s'_j} - q^{s'_i}} \cdot \prod_{1\leq i<j\leq d} \frac{q^{t_j} - q^{t_i}}{q^{t'_j} - q^{t'_i}}$$

Theorem (Shuffling Theorem)

For $U = \{s_1, s_2, \ldots, s_u\}$, $D = \{t_1, t_2, \ldots, t_d\}$, $U' = \{s'_1, s'_2, \ldots, s'_u\}$, $D' = \{t'_1, t'_2, \ldots, t'_d\}$ *of* $[x+y+n]$, *such that* $U \cup D = U' \cup D'$ *and* $U \cap D = U' \cap D' = \emptyset$

$$\frac{\mathrm{M}(H_{x,y}(U,D))}{\mathrm{M}(H_{x,y}(U',D'))} = \frac{\prod_{1\le i<j\le u} \frac{s_j - s_i}{j-i} \cdot \prod_{1\le i<j\le d} \frac{t_j - t_i}{j-i}}{\prod_{1\le i<j\le u} \frac{s'_j - s'_i}{j-i} \cdot \prod_{1\le i<j\le d} \frac{t'_j - t'_i}{j-i}}$$

All products can be expressed in terms of special Schur functions.

Schur functions

Theorem (Shuffling Theorem)

$$\frac{\mathrm{M}(H_{x,y}(U,D))}{\mathrm{M}(H_{x,y}(U',D'))} = \frac{\prod_{1\le i<j\le u} \frac{s_j - s_i}{j-i} \cdot \prod_{1\le i<j\le d} \frac{t_j - t_i}{j-i}}{\prod_{1\le i<j\le u} \frac{s'_j - s'_i}{j-i} \cdot \prod_{1\le i<j\le d} \frac{t'_j - t'_i}{j-i}}$$

- $\prod_{1\le i<j\le u} \frac{s_j - s_i}{j-i} = \mathbf{s}_{\lambda(s_1,\dots,s_u)}(1,1,\dots)$, where $\lambda(s_1,\dots,s_u) = (s_u - u + 1, s_{u-1} - u + 2, \dots, s_3 - 2, s_2 - 1, s_1)$

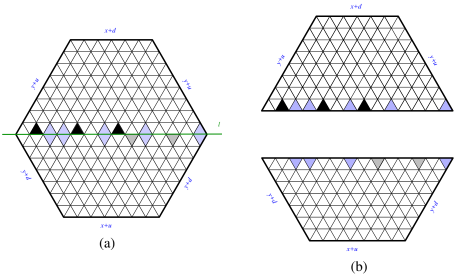
Schur functions

Theorem (Shuffling Theorem)

$$\frac{\mathrm{M}(H_{x,y}(U,D))}{\mathrm{M}(H_{x,y}(U',D'))} = \frac{\prod_{1\le i<j\le u} \frac{s_j - s_i}{j-i} \cdot \prod_{1\le i<j\le d} \frac{t_j - t_i}{j-i}}{\prod_{1\le i<j\le u} \frac{s'_j - s'_i}{j-i} \cdot \prod_{1\le i<j\le d} \frac{t'_j - t'_i}{j-i}}$$

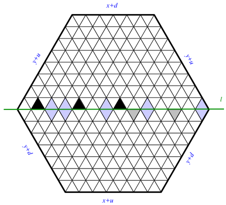
- $\prod_{1\le i<j\le u} \frac{s_j - s_i}{j-i} = \mathbf{s}_{\lambda(s_1,\dots,s_u)}(1,1,\dots)$, where $\lambda(s_1,\dots,s_u) = (s_u - u + 1, s_{u-1} - u + 2, \dots, s_3 - 2, s_2 - 1, s_1)$
- $RHS = \frac{\mathbf{s}_{\lambda(U)}(1,1,\dots)\mathbf{s}_{\lambda(D)}(1,1,\dots)}{\mathbf{s}_{\lambda(U')}(1,1,\dots)\mathbf{s}_{\lambda(D')}(1,1,\dots)}$

Schur functions

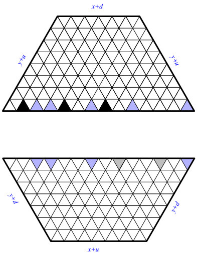


- $\mathrm{M}(H_{x,y}(U, D)) = \sum_{\substack{|S|=y \\ S \subseteq (U \cup D)^c}} \mathrm{M}(S_{x+d,y+u}(U \cup S))\mathrm{M}(S_{x+u,y+d}(D \cup S))$

Schur functions



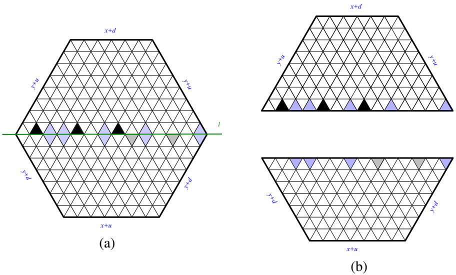
(a)



(b)

- $\mathrm{M}(H_{x,y}(U, D)) = \sum_{\substack{|S|=y \\ S\subseteq(U\cup D)^c}} \mathrm{M}(S_{x+d,y+u}(U \cup S))\mathrm{M}(S_{x+u,y+d}(D \cup S))$
- (Cohn–Larsen –Propp) $\mathrm{M}(S_{x+d,y+u}(U \cup S)) = \mathbf{s}_{\lambda(U\cup S)}(1, 1, \dots)$

Schur functions



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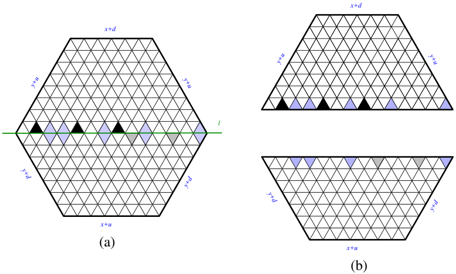
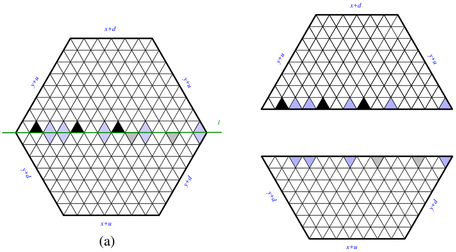
Schur functions



- $M(H_{x,y}(U,D)) = \sum_{\substack{|S|=y \\ S\subseteq(U\cup D)^c}} \mathbf{s}_{\lambda(U\cup S)}(1,1,\dots)\mathbf{s}_{\lambda(D\cup S)}(1,1,\dots)$

Schur functions



(a) (b)

- $\mathrm{M}(H_{x,y}(U, D)) = \sum_{\substack{|S|=y \\ S \subseteq (U \cup D)^c}} \mathbf{s}_{\lambda(U \cup S)}(1, 1, \ldots)\mathbf{s}_{\lambda(D \cup S)}(1, 1, \ldots)$
- $LHS = \frac{\mathrm{M}(H_{x,y}(U,D))}{\mathrm{M}(H_{x,y}(U',D'))} = \frac{\sum_{\substack{|S|=y \\ S \subseteq (U \cup D)^c}} \mathbf{s}_{\lambda(U \cup S)}(1,1,\ldots)\mathbf{s}_{\lambda(D \cup S)}(1,1,\ldots)}{\sum_{\substack{|S|=y \\ S \subseteq (U \cup D)^c}} \mathbf{s}_{\lambda(U' \cup S)}(1,1,\ldots)\mathbf{s}_{\lambda(D' \cup S)}(1,1,\ldots)}$

Schur functions

Theorem (Shuffling Theorem)

$$\frac{\sum_{\substack{|S|=y \\ S\subseteq(U\cup D)^c}} \mathbf{s}_{\lambda(U\cup S)}(1,1,\dots)\mathbf{s}_{\lambda(D\cup S)}(1,1,\dots)}{\sum_{\substack{|S|=y \\ S\subseteq(U\cup D)^c}} \mathbf{s}_{\lambda(U'\cup S)}(1,1,\dots)\mathbf{s}_{\lambda(D'\cup S)}(1,1,\dots)} = \frac{\mathbf{s}_{\lambda(U)}(1,1,\dots)\mathbf{s}_{\lambda(D)}(1,1,\dots)}{\mathbf{s}_{\lambda(U')}(1,1,\dots)\mathbf{s}_{\lambda(D')}(1,1,\dots)}$$

Schur functions

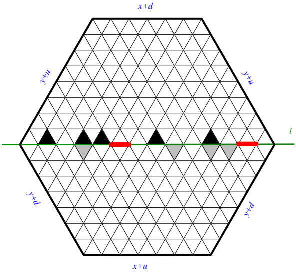
Theorem (*q*-Shuffling Theorem)

$$\frac{\sum_{\substack{|S|=y \\ S\subseteq (U\cup D)^c}} s_{\lambda(U\cup S)}(q,q^2,\dots)s_{\lambda(D\cup S)}(q,q^2,\dots)}{\sum_{\substack{|S|=y \\ S\subseteq (U\cup D)^c}} s_{\lambda(U'\cup S)}(q,q^2,\dots)s_{\lambda(D'\cup S)}(q,q^2,\dots)}$$

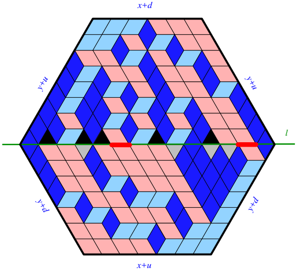
$$= q^C \frac{s_{\lambda(U)}(q,q^2,\dots)s_{\lambda(D)}(q,q^2,\dots)}{s_{\lambda(U')}(q,q^2,\dots)s_{\lambda(D')}(q,q^2,\dots)}$$

Question: Is there a Schur function identity behind this result?

Generalized Shuffling Theorem



(a)



(b)

- Upper position $U = \{s_1, s_2, \ldots, s_u\} \subset [x + y + n]$
- Lower position $D = \{t_1, t_2, \ldots, t_d\} \subset [x + y + n]$
- Position set of barriers $B = \{b_1, b_2, \ldots, b_k\}$
- Generalized doubly-dented hexagon: $H_{x,y}(U, D, B)$
- We now allow shuffling and flipping holes in $U \Delta D$.

Generalized Shuffling Theorem

Theorem

For $U = \{s_1, s_2, \dots, s_u\}$, $D = \{t_1, t_2, \dots, t_d\}$, $U' = \{s'_1, s'_2, \dots, s'_{u'}\}$, $D' = \{t'_1, t'_2, \dots, t'_{d'}\}$ *of* $[x+y+n]$, *such that* $U \cup D = U' \cup D'$ *and* $U \cap D = U' \cap D'$

$$\frac{\mathrm{M}(H_{x,y}(U,D,B))}{\mathrm{M}(H_{x,y}(U',D',B))} = \frac{\prod_{1\le i<j\le u} \frac{s_j - s_i}{j-i} \cdot \prod_{1\le i<j\le d} \frac{t_j - t_i}{j-i}}{\prod_{1\le i<j\le u'} \frac{s'_j - s'_i}{j-i} \cdot \prod_{1\le i<j\le d'} \frac{t'_j - t'_i}{j-i}} \times \frac{\mathrm{PP}(u,d,y)}{\mathrm{PP}(u',d',y)}$$

$$\mathrm{PP}(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

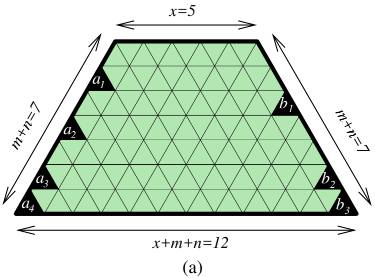
Generalized q-Shuffling Theorem

Theorem

$$\frac{\mathrm{M}_q(H_{x,y}(U,D,B))}{\mathrm{M}_q(H_{x,y}(U',D',B))} = q^{D} \cdot \frac{\prod_{1\le i<j\le u} \frac{q^{s_j}-q^{s_i}}{q^j-q^i} \cdot \prod_{1\le i<j\le d} \frac{q^{t_j}-q^{t_i}}{q^j-q^i}}{\prod_{1\le i<j\le u'} \frac{q^{s'_j}-q^{s'_i}}{q^j-q^i} \cdot \prod_{1\le i<j\le d'} \frac{q^{t'_j}-q^{t'_i}}{q^j-q^i}} \times \frac{\mathrm{PP}_q(u,d,y)}{\mathrm{PP}_q(u',d',y)}$$

$$\mathrm{PP}_q(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

Semi-hexagon with dents



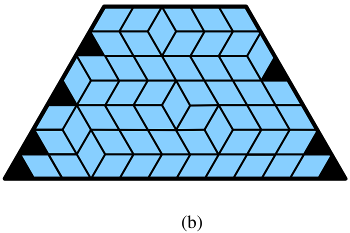
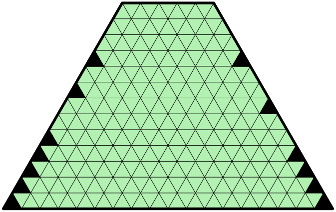
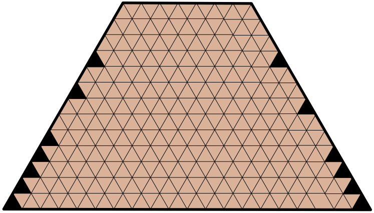
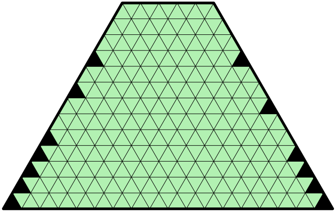


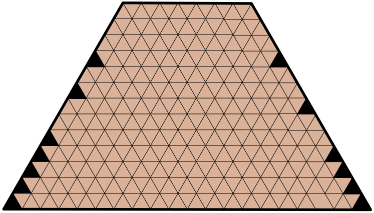
Figure: The region $S_x((a_i)_{i=1}^m; (b_j)_{j=1}^n) = S_5((2,4,6,7); (3,6,7))$ and a tiling of its.





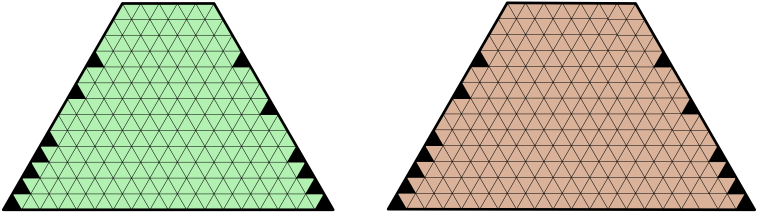
- Tiling number of $S_5((4,6,9,10,11,12,13);(4,7,10,11,12,13))$: $2^6 \cdot 7^3 \cdot 11 \cdot 13^4 \cdot 17 \cdot 2683$





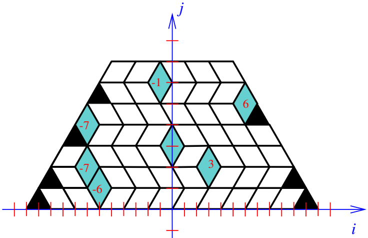
- Tiling number of $S_5((4,6,9,10,11,12,13);(4,7,10,11,12,13))$: $2^6 \cdot 7^3 \cdot 11 \cdot 13^4 \cdot 17 \cdot 2683$
- Tiling number of $S_7((4,6,9,10,11,12,13);(4,7,10,11,12,13))$ is: $2^8 \cdot 3^4 \cdot 5 \cdot 13^3 \cdot 17^3 \cdot 19 \cdot 2683$

Shuffling Phenomenon

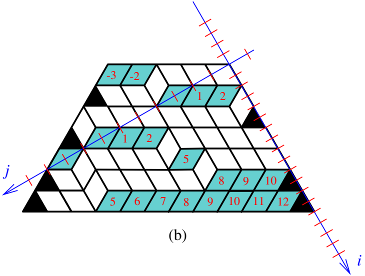


- Tiling number of $S_5((4,6,9,10,11,12,13);(4,7,10,11,12,13))$: $2^6 \cdot 7^3 \cdot 11 \cdot 13^4 \cdot 17 \cdot 2683$
- Tiling number of $S_7((4,6,9,10,11,12,13);(4,7,10,11,12,13))$ is: $2^8 \cdot 3^4 \cdot 5 \cdot 13^3 \cdot 17^3 \cdot 19 \cdot 2683$
- The ratio of tiling numbers: $2^{-2} \cdot 3^{-4} \cdot 5^{-1} \cdot 7^3 \cdot 11 \cdot 13 \cdot 17^{-2} \cdot 19^{-1}$

How to assign weight to tilings



(a)



(b)

(A. Borodin–V. Gorin –E. Rains) $w(n) = \frac{Xq^n + Yq^{-n}}{2}$

Shuffling Theorem for symmetric tilings

Theorem (L. 2020)

$$\frac{\mathrm{M}_{X,Y,q}(S_x((a_i)_{i=1}^m;(b_j)_{j=1}^n))}{\mathrm{M}_{X,Y,q}(S_y((a_i)_{i=1}^m;(b_j)_{j=1}^n))}$$

$$= q^{(y-x)\left(\sum_{i=1}^m a_i+\sum_{j=1}^n b_j-\frac{(m+n)(m+n+1)}{2}\right)}\frac{\mathrm{PP}_{q^2}(y,m,n)}{\mathrm{PP}_{q^2}(x,m,n)}$$

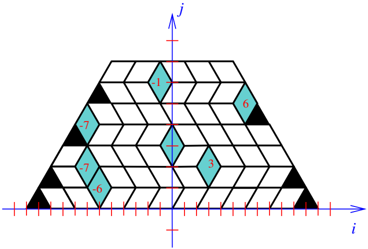
$$\times\prod_{i=1}^m\frac{(-q^{2(x+i)};q^2)_{a_i-i}}{(-q^{2(y+i)};q^2)_{a_i-i}}\prod_{j=1}^n\frac{(-q^{2(x+j)};q^2)_{b_j-j}}{(-q^{2(y+j)};q^2)_{b_j-j}},$$

where

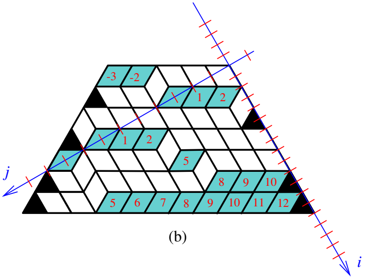
$$\mathrm{PP}_q(a,b,c)=\prod_{i=1}^a\prod_{j=1}^b\prod_{k=1}^c\frac{q^{i+j+k-1}-1}{q^{i+j+k-2}-1}$$

$$(x;q)_n := (1+x)(1+xq)(1+xq^2)\cdots(1+xq^{n-1}).$$

How to assign weight to tilings



(a)



(b)

$$w(n) = \frac{Xq^{n} + Yq^{-n}}{2}$$

Shuffling Theorem for symmetric tilings

Theorem (L. 2020)

$$\frac{\mathrm{M}_{X,Y,q}(S'_x((a_i)_{i=1}^m;(b_j)_{j=1}^n))}{\mathrm{M}_{X,Y,q}(S'_y((a_i)_{i=1}^m;(b_j)_{j=1}^n))} = q^C \frac{\mathrm{PP}_{q^2}(y,m,n)}{\mathrm{PP}_{q^2}(x,m,n)} \prod_{j=1}^{n}\prod_{i=1}^{y-x}(X^2+q^{2(x+i-b_j)}XY) \times \prod_{i=1}^{m}\frac{(-q^{2(x+i)};q^2)_{a_i-i}}{(-q^{2(y+i)};q^2)_{a_i-i}} \prod_{j=1}^{n}\frac{(-q^{2(x+j)};q^2)_{b_j-j}}{(-q^{2(y+j)};q^2)_{b_j-j}}.$$

Daniel Condon independently proved the unweighted case $(X = Y = q = 1)$.

Quartered Hexagon with Dents

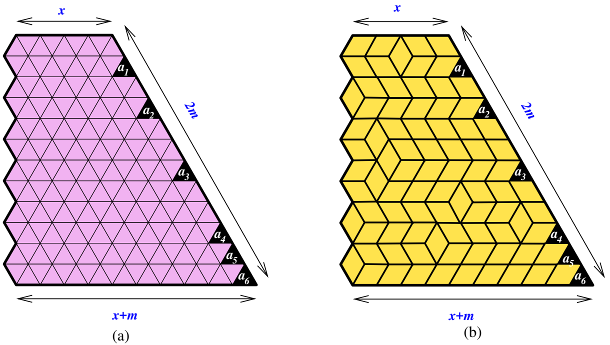


Figure: The region $Q_x((a_i)_{i=1}^m) = Q_4(2, 4, 7, 10, 11, 12)$ and a tiling of its.

Second Shuffling Theorem

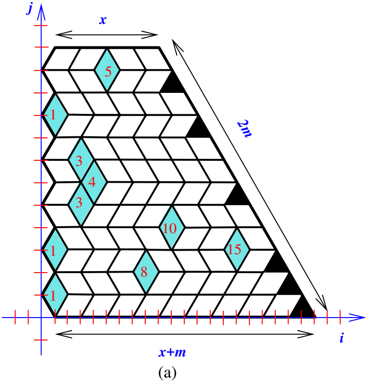


Figure: Two ways to assign weight to tilings of the quartered hexagon.

$$w_0(n) = \frac{q^n + q^{-n}}{2}$$

$$(X = Y = 1)$$

Second Shuffling Theorem

Theorem (L. 2020)

$$\frac{\mathrm{M}_q(Q_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q_y((a_i)_{i=1}^m))} = q^{2(y-x)(\sum_{i=1}^m a_i - m^2)} \prod_{i=1}^{m} \frac{(-q^{2(2y+a_i+1)};q^2)_{2i-a_i-1}}{(-q^{2(2x+a_i+1)};q^2)_{2i-a_i-1}}$$

Second Shuffling Theorem

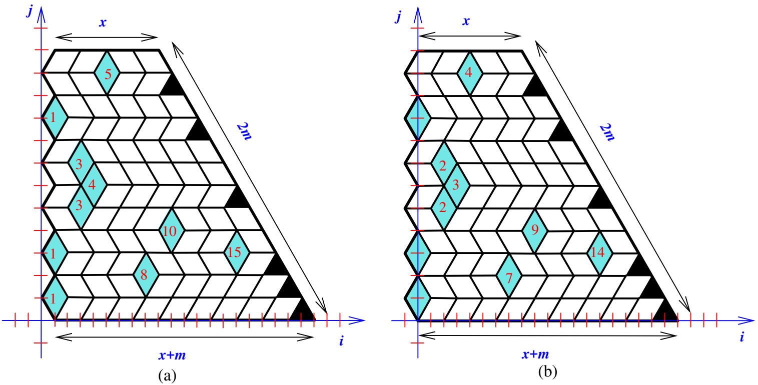


Figure: Two ways to assign weight to tilings of the quartered hexagon.

$$w_0(n) = \frac{q^n + q^{-n}}{2}$$

Second Shuffling Theorem

Theorem (L. 2020)

$$\frac{\mathrm{M}_q(Q'_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q'_y((a_i)_{i=1}^m))} = q^{2(y-x)(\sum_{i=1}^m a_i - m^2)} \prod_{i=1}^{m} \frac{(-q^{2(2y+a_i)}; q^2)_{2i-a_i-1}}{(-q^{2(2x+a_i)}; q^2)_{2i-a_i-1}}$$

Comparing two formulas

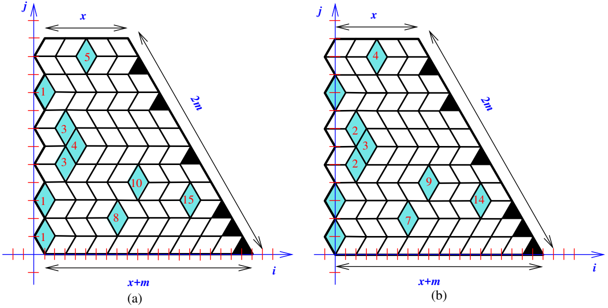


Figure: Two ways to assign weight to tilings of the quartered hexagon.

- $\dfrac{\mathrm{M}_q(Q'_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q'_y((a_i)_{i=1}^m))} = \dfrac{\mathrm{M}_q(Q_{x-1/2}((a_i)_{i=1}^m))}{\mathrm{M}_q(Q_{y-1/2}((a_i)_{i=1}^m))}$

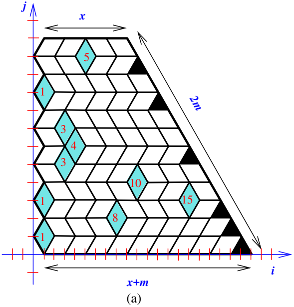


Figure: Two ways to assign weight to tilings of the quartered hexagon.

- $\dfrac{\mathrm{M}_q(Q'_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q'_y((a_i)_{i=1}^m))} = \dfrac{\mathrm{M}_q(Q_{x-1/2}((a_i)_{i=1}^m))}{\mathrm{M}_q(Q_{y-1/2}((a_i)_{i=1}^m))}$
- But it is wrong!!!

Comparing two formulas

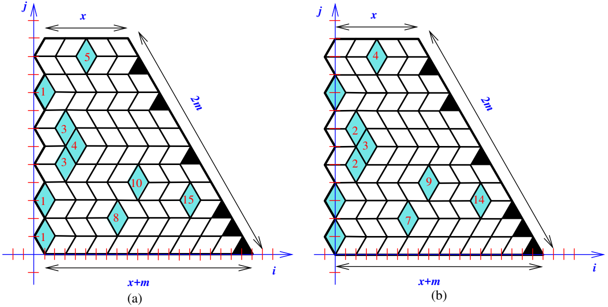
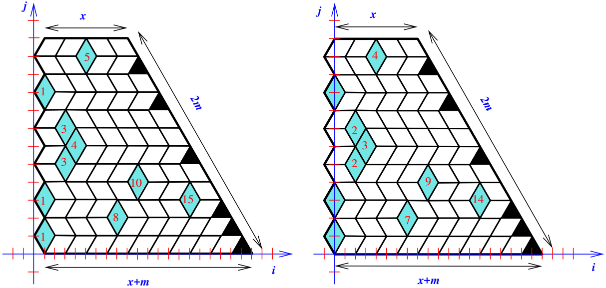


Figure: Two ways to assign weight to tilings of the quartered hexagon.

- $\dfrac{\mathrm{M}_q(Q'_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q'_y((a_i)_{i=1}^m))} = \dfrac{\mathrm{M}_q(Q_{x-1/2}((a_i)_{i=1}^m))}{\mathrm{M}_q(Q_{y-1/2}((a_i)_{i=1}^m))}$
- But it is wrong!!!
- Our regions are *not* defined when its side-lengths are half-integers.

Reciprocity-like phenomenon



(a) (b)

Figure: Two ways to assign weight to tilings of the quartered hexagon.

- Let $f_{x,y}((a_i)_{i=1}^m) = \frac{\mathrm{M}_q(Q_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q_y((a_i)_{i=1}^m))}$ and $g_{x,y}((a_i)_{i=1}^m) = \frac{\mathrm{M}_q(Q'_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q'_y((a_i)_{i=1}^m))}$.

Reciprocity-like phenomenon

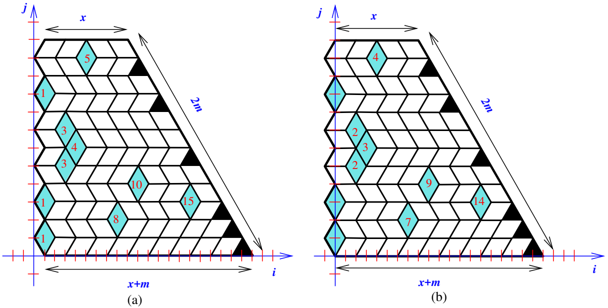


Figure: Two ways to assign weight to tilings of the quartered hexagon.

- Let $f_{x,y}((a_i)_{i=1}^m) = \frac{\mathrm{M}_q(Q_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q_y((a_i)_{i=1}^m))}$ and $g_{x,y}((a_i)_{i=1}^m) = \frac{\mathrm{M}_q(Q'_x((a_i)_{i=1}^m))}{\mathrm{M}_q(Q'_y((a_i)_{i=1}^m))}$.
- Then $g_{x,y}((a_i)_{i=1}^m) = f_{x-1/2,y-1/2}((a_i)_{i=1}^m)$.

Reciprocity-like phenomenon

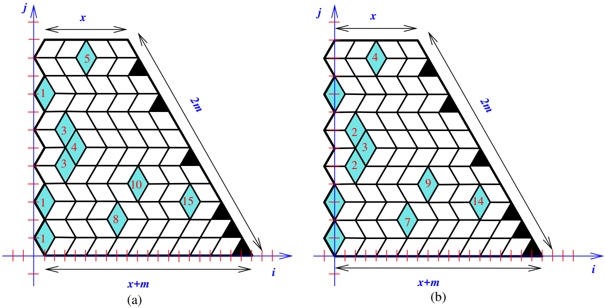


Figure: Two ways to assign weight to tilings of the quartered hexagon.

- Let $f_{x,y}((a_i)_{i=1}^m) = \frac{M_q(Q_x((a_i)_{i=1}^m))}{M_q(Q_y((a_i)_{i=1}^m))}$ and $g_{x,y}((a_i)_{i=1}^m) = \frac{M_q(Q'_x((a_i)_{i=1}^m))}{M_q(Q'_y((a_i)_{i=1}^m))}$.
- Then $g_{x,y}((a_i)_{i=1}^m) = f_{x-1/2,y-1/2}((a_i)_{i=1}^m)$.
- It reminds us to the Reciprocity Phenomenon.

Reciprocity-like phenomenon

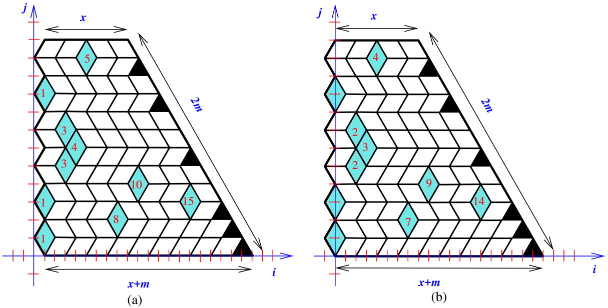
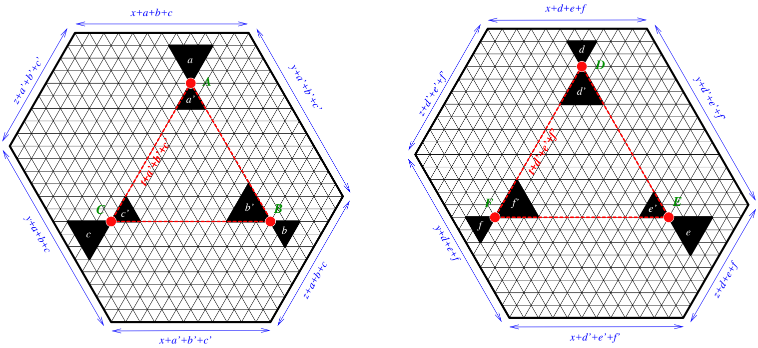


Figure: Two ways to assign weight to tilings of the quartered hexagon.

- Let $f_{x,y}((a_i)_{i=1}^m) = \frac{\mathrm{M}(Q_x((a_i)_{i=1}^m))}{\mathrm{M}(Q_y((a_i)_{i=1}^m))}$ and $g_{x,y}((a_i)_{i=1}^m) = \frac{\mathrm{M}(Q'_x((a_i)_{i=1}^m))}{\mathrm{M}(Q'_y((a_i)_{i=1}^m))}$.
- Then $g_{x,y}((a_i)_{i=1}^m) = f_{x-1/2,y-1/2}((a_i)_{i=1}^m)$.
- **Question:** Is there a combinatorial explanation for this?

Hexagons with a triad of bowties removed



(a) (b)

Figure: Two 'sibling' regions: $R = R^{\Delta}_{x,y,z}(a, a', b, b', c, c')$ and $R' = R^{\Delta'}_{x,y,z}(d, d', e, e', f, f')$.

Preparation for the formula Shuffling Theorem

Define the "hyperfactorial" is defined by

$$\mathrm{H}(n) = 0! \cdot 1! \cdot 2! \cdots (n-1)!.$$

$$w := \frac{\mathrm{H}(s)^4\, \mathrm{H}(a)\, \mathrm{H}(b)\, \mathrm{H}(c)\, \mathrm{H}(a')\, \mathrm{H}(b')\, \mathrm{H}(c')}{\mathrm{H}(s+a)\, \mathrm{H}(s+b)\, \mathrm{H}(s+c)\, \mathrm{H}(s-a')\, \mathrm{H}(s-b')\, \mathrm{H}(s-c')}$$

where $s = t + a' + b' + c'$.

$$w' := \frac{\mathrm{H}(s')^4\, \mathrm{H}(d)\, \mathrm{H}(e)\, \mathrm{H}(f)\, \mathrm{H}(d')\, \mathrm{H}(e')\, \mathrm{H}(f')}{\mathrm{H}(s'+d)\, \mathrm{H}(s'+e)\, \mathrm{H}(s'+f)\, \mathrm{H}(s'-d')\, \mathrm{H}(s'-e')\, \mathrm{H}(s'-f')}$$

where $s' = t + d' + e' + f'$.

Third Shuffling Theorem

Theorem (Ciucu–L.–Rohatgi 2019)

$$\frac{\mathrm{M}(R^{\Delta}_{x,y,z}(a,a',b,b',c,c'))}{\mathrm{M}(R^{\Delta'}_{x,y,z}(d,d',e,e',f,f'))} = \frac{w \cdot \frac{k_A(R)k_B(R)k_C(R)}{k_{BC}(R)k_{CA}(R)k_{AB}(R)}}{w' \cdot \frac{k_D(R')k_E(R')k_F(R')}{k_{EF}(R')k_{FD}(R')k_{DE}(R')}}$$

where

$$K_A(R) = \mathrm{H}(d(A,N))\mathrm{H}(d(A,S))$$

$$K_B(R) = \mathrm{H}(d(B,NE))\mathrm{H}(d(B,SW))$$

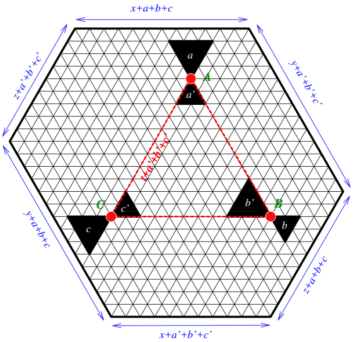
$$K_C(R) = \mathrm{H}(d(C,NW))\mathrm{H}(d(C,SE))$$

$$K_{BC}(R) = \mathrm{H}(d(BC,N))\mathrm{H}(d(BC,S))$$

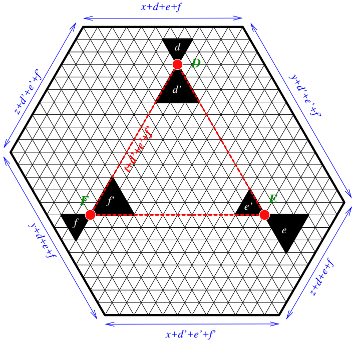
$$K_{AC}(R) = \mathrm{H}(d(AC,NE))\mathrm{H}(d(AC,SW))$$

$$K_{AB}(R) = \mathrm{H}(d(AB,NW))\mathrm{H}(d(AB,SE))$$

Third shuffling Theorem

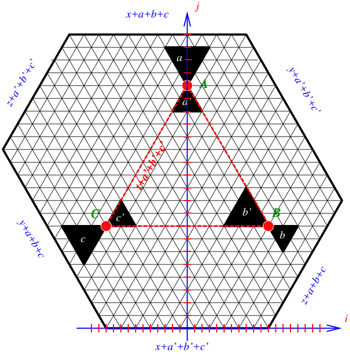


(a)



(b)

A q-analog



(a) (b)

$$\color{red}{w(n)} = \frac{\color{blue}{X}q^{\color{red}{n}} + \color{green}{Y}q^{-\color{red}{n}}}{2}$$

A q-analog

Conjecture

The ratio of tiling generating functions

$$\frac{\mathrm{M}_{X,Y,q}(R^{\Delta}_{x,y,z}(a,a',b,b',c,c'))}{\mathrm{M}_{X,Y,q}(R^{\Delta'}_{x,y,z}(d,d',e,e',f,f'))}$$

is always given by a simple product formula.

Thank you!

Email: tlai3@unl.edu
Website: http://www.math.unl.edu/∼tlai3/