Dimers and Webs

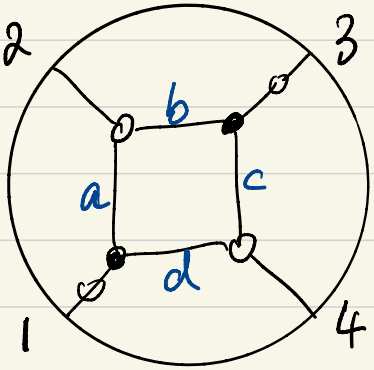
Thomas Lam
University of Michigan

August 2020

arXiv: 1404.3317 } Lecture 1
arXiv: 1506.00603 }

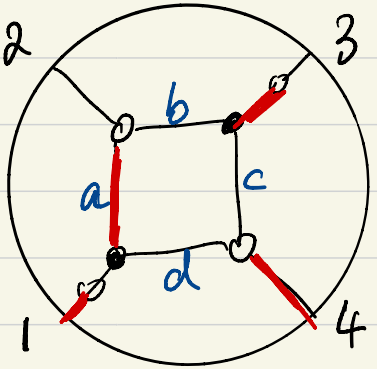
arXiv: 1705.09424 (j. C. Fraser and I. Le) Lecture 2

$N =$



planar bipartite graph

{boundary vertices are black and degree one}



$\partial(\pi) = \{1,4\}$

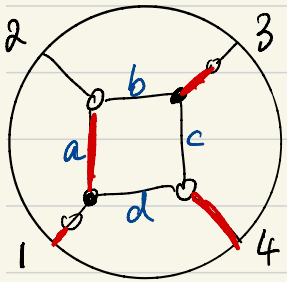
$wt(\pi) = a$

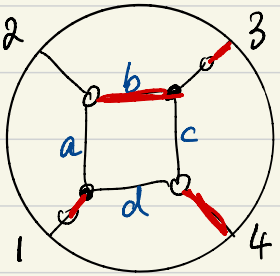
π = almost perfect matching (uses all interior vertices once.

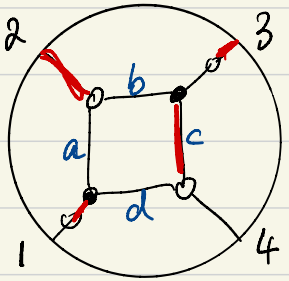
$$\Delta_I(N) = \sum_{\pi:\, \partial(\pi) = I} wt(\pi)$$

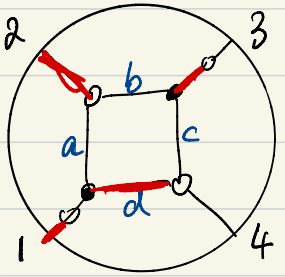
$|I| = k :=$ # interior white − # interior black

Example of diner generating function:

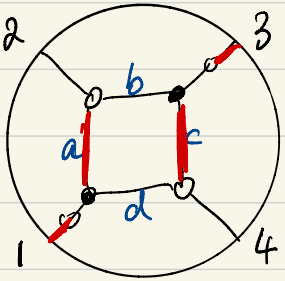


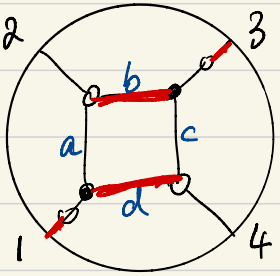


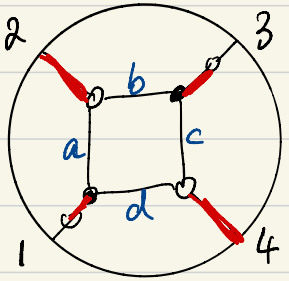




wt	a	b	c	d
∂	14	34	23	12







ac	bd	1
13	13	24

$$\Delta_{13}(N) = ac + bd$$

Theorem 1

[Kuo
Postnikov, Talaska, Postnikov – Speyer Williams
L.]

$(\Delta_I(N))_{I \in \binom{[n]}{k}}$ defines a point in $Gr(k,n)$

ie. satisfies the Plücker relation

$$\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1, i_2, \dots i_{k-1}, j_s} \Delta_{j_1, \dots \hat{j_s} \dots j_{k+1}} = 0$$

e.g. $k=2$ $i_1 = 1$ $j_1, j_2, j_3 = 2, 3, 4$.

$$-\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23} = 0.$$

$$-d \cdot b + (ac+bd) \cdot 1 - a \cdot c = 0$$

Proof using "webs"

Definition: A subgraph $\Sigma \subset N$ is **2-weblike** if it is a union of connected components:

(a) paths between boundary vertices
(b) interior cycles
(c) single edges (called "doubled edge")

such that every interior vertex is used.

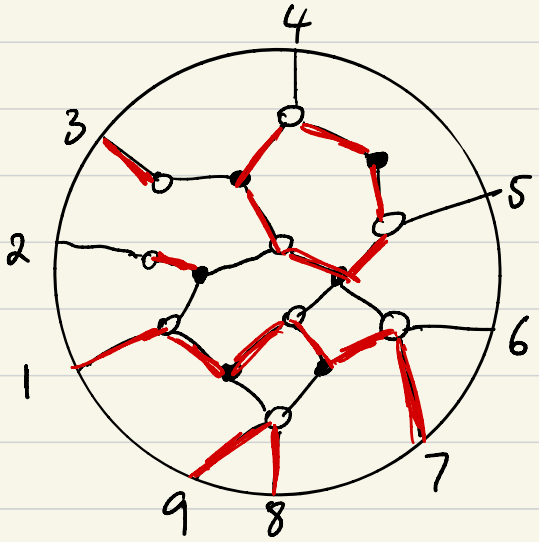
$$F_{\tau,T}(N) := \sum_{\Sigma} wt(\Sigma)$$

Temperley-Lieb invariant.

[L., cf. Rhoades-Skandera, cf. Kenyon-Wilson]

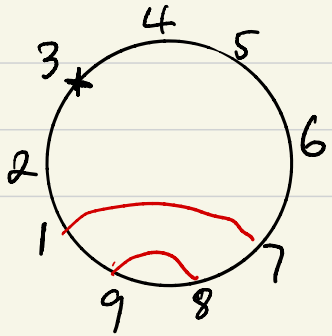
τ = partial non crossing matching
T = boundary vertices used in doubled edges

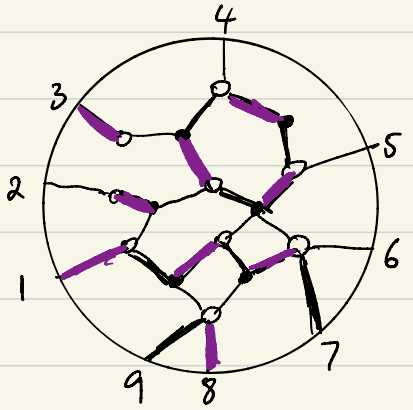
$$wt(\Sigma) = \prod_{e \in \text{path or cycle}} wt(e) \cdot \prod_{e' \in \text{doubled edge}} wt(e')^2 \cdot 2^{\#\text{cycle}}$$



Σ

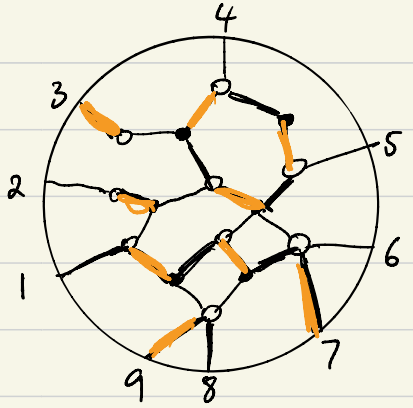
$\tau, T =$





$I = 138$

Double dimer



$J = 379$

Definition $I, J \in \binom{[n]}{k}$ is **compatible** with (τ, T) if:

- $T = I \cap J$
- each chord of τ matches $i \in I - J$ with $j \in J - I$

Theorem 2 [L., cf. Kenyon-Wilson, Rhoades-Skandera, ...]

$$\Delta_I(N)\,\Delta_J(N) = \sum_{\substack{(\tau,T) \\ \text{comp. with } (I,J)}} F_{\tau,T}(N)$$

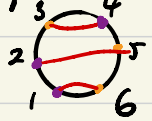
$R(k,n) := \mathbb{C}[Gr(k,n)]$

$R(k,n) = \bigoplus_{d \geq 0} R(k,n)_d$

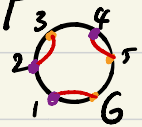
$R(k,n)_1 = \bigoplus \mathbb{C} \cdot \Delta_I$

$R(k,n)_2 = \bigoplus \mathbb{C} \cdot F_{\tau,T}$

e.g. $\Delta_{124}\Delta_{356} =$



$+$



- $F_{\tau,T}$ is a well-defined quadratic function on $\widehat{Gr}(k,n)$
- forms a cyclically invariant basis
- coincides with "dual canonical basis", better than "standard monomial basis"

$\Delta_{S_1}\Delta_{S_2}$ $\boxed{S_1 | S_2}$ semi standard

- Theorem 2 $\Rightarrow$ Theorem 1

Expand $\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1, i_2, \ldots i_{k-1}, j_s} \Delta_{j_1, \ldots \hat{j_s} \ldots j_{k+1}}$ in terms of $F_{c,\bar{1}}$

and do a sign-reversing involution.

- $\Delta_I \Delta_J \leq \Delta_{\min(I,J)} \Delta_{\max(I,J)} \leq \Delta_{\text{sort}_1(I,J)} \Delta_{\text{sort}_2(I,J)}$

e.g. $\Delta_{13567} \Delta_{23489} \leq \Delta_{13467} \Delta_{23589} \leq \Delta_{13468} \Delta_{23579}$

sort = 1233456789

[cf. Rhoades-Skandera
Fabev-Postnikov
L.-Postnikov-Pylyavskyy]

- $\mathcal{M}(N) := \left\{ I \in \binom{[n]}{k} \;\middle|\; \Delta_I(N) \neq 0 \right\}$ matroid of N.

$I, J \in \mathcal{M} \Rightarrow sort_1, sort_2 \in \mathcal{M}$ [L.-Postnikov]

"Sort-closed matroid"

- $\mathcal{M}^{(2)}(N) := \{ \tau, T \mid F_{\tau,T}(N) \neq 0 \}$ "quadratic matroid"

- any positive quadratic function, homogeneous with respect to torus action, is a positive linear combination of $F_{\tau,T}$
- applications to dimers?

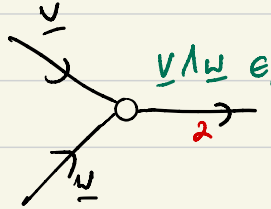
Based on joint work with C. Fraser and I. Le

Webs

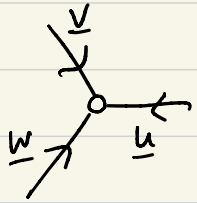
$k \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

$$\mathbb{C}[Gr(k,n)] = R(k,n) = SL_k \text{ invariant functions on } \underbrace{\mathbb{C}^k \times \cdots \times \mathbb{C}^k}_{n} = \mathbb{C}[\Delta_I] / \text{relations}$$

$k=3$

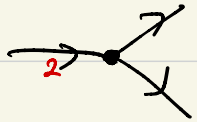


$\underline{v} \wedge \underline{w} \in \wedge^2(\mathbb{C}^3)$

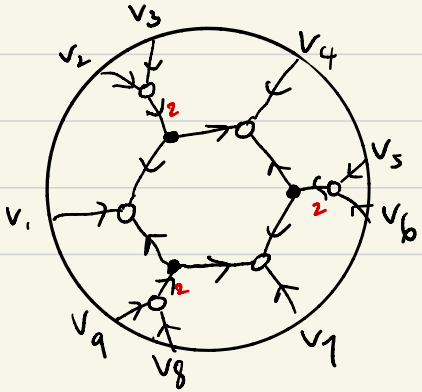


$\det(\underline{v}|\underline{w}|\underline{u})$

$\wedge^2\mathbb{C}^3 \hookrightarrow \mathbb{C}^3 \otimes \mathbb{C}^3$



[Reshetikhin–Turaev
Kuperberg]



$\in R(3,9)_3$

$v_i \in \mathbb{C}^3$

Clusters and webs

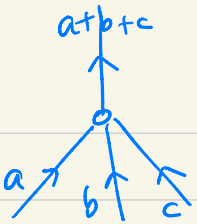
[Fomin–Pylyavskyy]

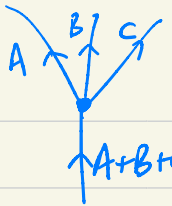
Cluster structure of $\mathbb{C}[Gr(3,n)]$ (+ more) can be described in terms of web invariants:

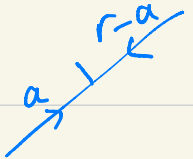
- initial cluster, quivers, coefficients
- cluster variables are indecomp. web invariants
- monomials ↔ tensor invariant
- compatibility

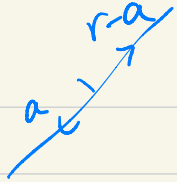
Conjecture [Fraser Gr(3,9)]

SL(r)-webs

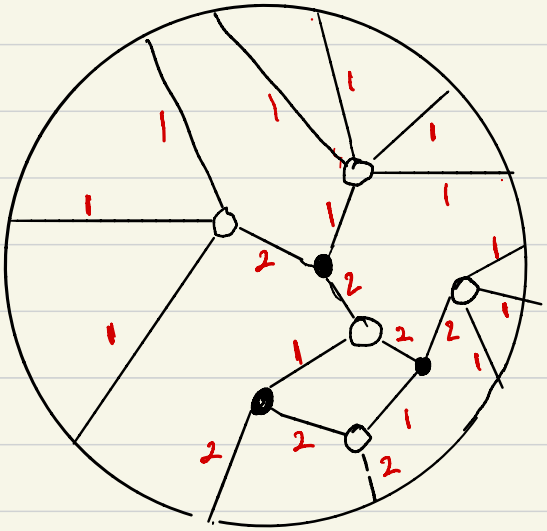




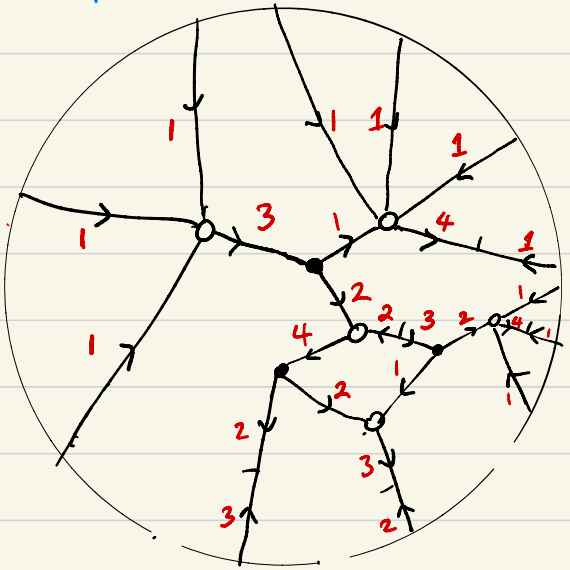




r=5

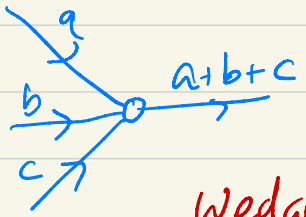


untagged SL(5)-web



$\lambda = (1,1,1,\ldots,1,2,3)$

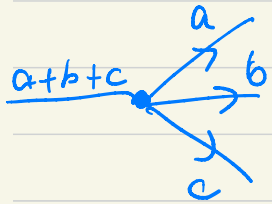
tagged SL(5)-web



Wedge

$$\Lambda^a(\mathbb{C}^r)\otimes\Lambda^b(\mathbb{C}^r)\otimes\Lambda^c(\mathbb{C}^r)\to\Lambda^{a+b+c}(\mathbb{C}^r)$$

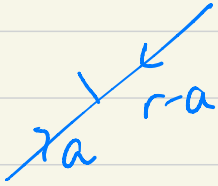
$$x_1\otimes x_2\otimes x_3\longmapsto x_1\wedge x_2\wedge x_3$$



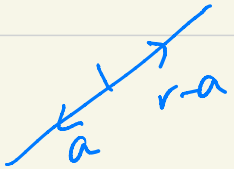
shuffle

$$\Lambda^{a+b+c}(\mathbb{C}^r)\to\Lambda^a(\mathbb{C}^r)\otimes\Lambda^b(\mathbb{C}^r)\otimes\Lambda^c(\mathbb{C}^r)$$

$$x_1\wedge\dots\wedge x_{a+b+c}\longrightarrow\sum\pm(x_{i_1}\wedge\dots\wedge x_{i_a})\otimes(\quad)\otimes($$



$$\Lambda^a(\mathbb{C}^r)\otimes\Lambda^{r-a}(\mathbb{C}^r)\longrightarrow\Lambda^r(\mathbb{C}^r)\cong\mathbb{C}$$



$$\Lambda^r(\mathbb{C}^r)\to\Lambda^a(\mathbb{C}^r)\otimes\Lambda^{r-a}(\mathbb{C}^r)$$

Lemma Any tagging $\hat{w}$ of a web W produces the same function up to sign.

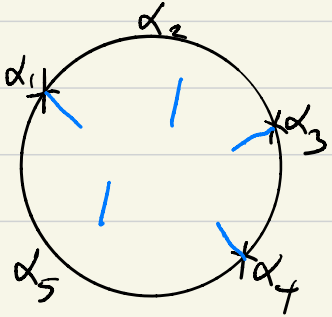
$$W_\lambda(r) = \mathrm{Hom}_{SL(r)}\left(\bigotimes_{i=1}^{n} \Lambda^{\lambda_i}\mathbb{C}^r, \mathbb{C}\right) \qquad \lambda=(\lambda_1,\dots,\lambda_n)$$

$$W(r,n) := \mathrm{Hom}_{SL(r)}\left(\bigotimes_{i=1}^{n} \mathbb{C}^r, \mathbb{C}\right)$$

are spanned by appropriate $\hat{w}$.

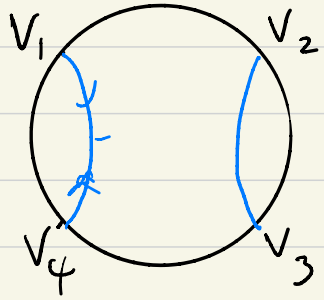
Ex. $r=1$

$\alpha_i \in \mathbb{C}$



$$\hat{w} = \pm \alpha_1\alpha_3\alpha_4$$

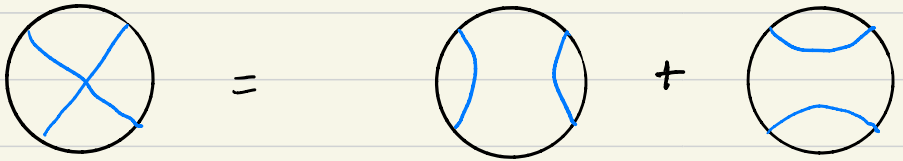
$r=2$ $\qquad v_i \in \mathbb{C}^2$



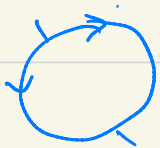
$= \pm \det(v_1|v_4) \det(v_2|v_3)$

basis of $W(2,2n)$ = noncrossing matchings on $[2n]$

[Rumer-Teller-Weyl]

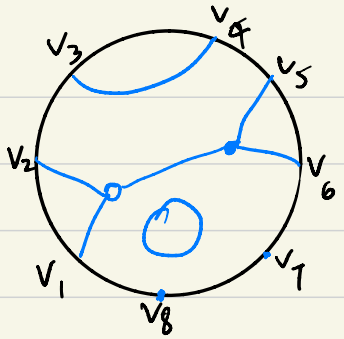


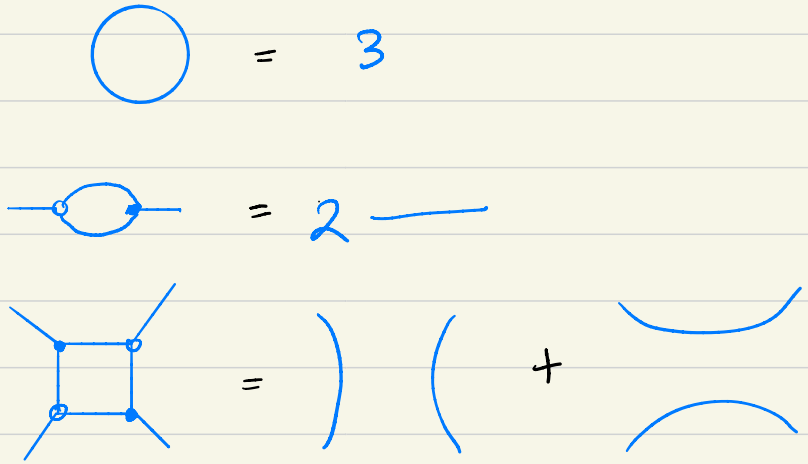
$= \pm 2 = \text{trace}(\text{Id}: \mathbb{C}^2 \to \mathbb{C}^2)$



$r = 3$

e.g.





[Kuperberg] basis = irreducible / elliptic webs

ie. no contractible loops, no square faces

Definition An *r-webLike subgraph* $W \subset N$ is a subgraph using all vertices of N, with each edge labeled by a multiplicity $m(e)$, such that

$$\sum_{e \text{ incident to } v} m(e) = r \qquad \text{for interior } v$$

$$wt(W) = \prod_e wt(e)^{m(e)}$$

$\lambda(W)$ = boundary multiplicities.

Each such W gives an untagged $SL(r)$-web and a tensor invariant $\underline{W}$ ($= \pm \hat{W}$ for some tagging)

$$\mathrm{Web}_r(N;\lambda) := \sum_{\substack{WCG \\ \lambda(W)=\lambda}} \mathrm{wt}(W)\,\underline{W} \quad \in \mathcal{W}_\lambda(r)$$

- $r=1$ $\quad \lambda \longleftrightarrow I \quad$ $\mathrm{Web}_1(N;\lambda) = \Delta_I(N)$

Theorem 1 [Fraser, L., Le] If $\Delta_I(N) = \Delta_I(N')$ for all I, then

$$\mathrm{Web}_r(N;\lambda) = \mathrm{Web}_r(N';\lambda) \quad \in \mathcal{W}_\lambda(r)$$

for all r, λ.

$\mathbb{C}[Gr(k,n)]_\lambda$ = be the λ-weight component.

e.g. $\Delta_{124}\Delta_{246}$ has weight $(1,2,0,2,0,1)$

Theorem 2 $Imm: W_1(U)^* \longrightarrow \mathbb{C}[Gr(k,n)]_\lambda$

$$\varphi \longmapsto \left(N \longmapsto \varphi(Web_r(N;\lambda)) \right)$$

is an isomorphism.

Theorem 3 $\Delta_{I_1}(N)\Delta_{I_2}(N)\dots\Delta_{I_r}(N) = sign(S)\, Web_r(N;\lambda)\big|_{e_S}$

where

$S = (S(1),\dots,S(n))$ $S(j) \subset \binom{[r]}{\lambda_j}$

I, S related by

$I_i = \{ j \in [n] : i \in S(j) \}$

i.e. transpose as 01-matrices.

$e_S := e_{S(1)} \otimes \dots \otimes e_{S(n)} \in \wedge^{\lambda_1}(\mathbb{C}^r) \otimes \dots \otimes \wedge^{\lambda_n}(\mathbb{C}^r)$

$(e_{124} = e_1 \wedge e_2 \wedge e_4)$

Remarks

- $r=2$ $W_\lambda(2)$ has noncrossing matching basis:

e.g.



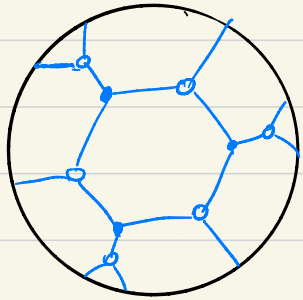
$\in W_{(1,1,1,1,1,1)}(2) = W(2,6) \xleftrightarrow{\text{dual}} \mathbb{C}[Gr(3,6)]_{(1,1,1,\ldots)}$

$\mathrm{Imm}(\text{dual basis element}) = F_{\tau,T}$.

- $r=3$

[Kuperberg] $W_1(3)$ has basis of irreducible/elliptic webs.





no square faces!

Imm(dual basis element) = "web immanant" [L. cf. Pylyavskyy]

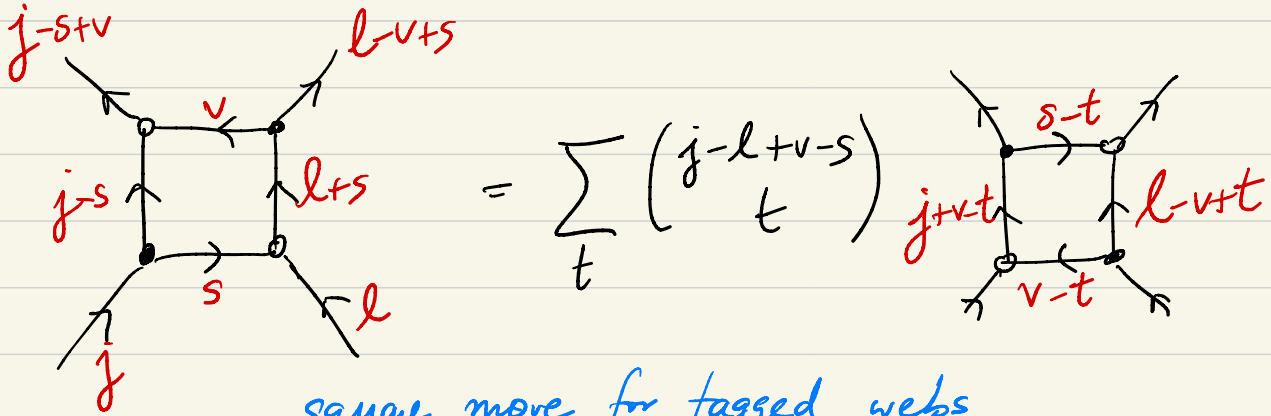
- For $r>3$, no generally agreed upon web basis is known. There is a dual canonical basis.

- generators for relations b/w webs are known

[Kuperberg, ..., Cautis-Kamnitzer-Morrison]

Theorem Under $\mathrm{Web}_\Gamma(N;\lambda) = \mathrm{Web}_{\Gamma'}(N';\lambda')$

local moves for planar bipatite graphs $\Rightarrow$ web relations



square move for tagged webs

Further directions

- Quantization?
- Positivity of $\underline{W}$ as function on $Gr(r,n)$?
- Applications to dimers?
- Relation Imm to cluster structure?