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Spaces of quasiperiodic sequences

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Central object of study

Spaces of quasiperiodic sequences and their moduli $\mathrm{QpGr}(\pi)$.

Summary of results

To a reduced plabic graph with positroid π, we construct a map

$$\beta : (\mathbb{K}^\times)^F \longrightarrow \mathrm{QpGr}(\pi)$$

This map is a toric chart in a (partial) Y-type cluster structure on $\mathrm{QpGr}(\pi)$ which makes it into the dual cluster variety to $\widehat{\mathrm{Gr}}(\pi)$.

Some general notation

- $\mathbb{K}$ is a field, which we fix throughout.
- π is a positroid (or an equivalent combinatorial object).
- $\mathrm{Gr}(\pi)$ is the corresponding (open) positroid variety.
- $\widehat{\mathrm{Gr}}(\pi)$ is the Plücker cone over $\mathrm{Gr}(\pi)$.

Quasiperiodic sequences and spaces

For us, a sequence is an element of $\mathbb{K}^{\mathbb{Z}}$; i.e. a bi-infinite list in $\mathbb{K}$.

Defn: A quasiperiodic sequence

A sequence v in $\mathbb{K}^{\mathbb{Z}}$ is **quasiperiodic** if there exists $n \in \mathbb{N}$ and $\lambda \in \mathbb{K}^{\times}$ such that $v_{a+n} = \lambda v_a$ for all a.

We write '(n, λ)-quasiperiodic' when we want to fix n and λ.

Example: Three $(4, 2)$-quasiperiodic sequences

$\cdots$	0	.5	$-.5$	-1	0	1	-1	-2	0	2	-2	-4	$\cdots$
$\cdots$	1.5	2.5	.5	-2	3	5	1	-4	6	10	2	-8	$\cdots$
$\cdots$	1	.5	1.5	1	2	1	3	2	4	2	6	4	$\cdots$

Defn: A quasiperiodic space

A subspace of $\mathbb{K}^{\mathbb{Z}}$ is **quasiperiodic** if there exists $n \in \mathbb{N}$ and $\lambda \in \mathbb{K}^{\times}$ such that every element is (n, λ)-quasiperiodic.

Examples

- The span of a quasiperiodic sequence.
- The space of solutions to the linear recurrence

$$x_i = x_{i-1} - x_{i-2} \qquad \text{(odd } i\text{)}$$
$$x_i = -x_{i-1} + 2x_{i-2} \qquad \text{(even } i\text{)}$$

Intuitively, (n, λ)-qp objects in $\mathbb{K}^{\mathbb{Z}}$ are equivalent to objects in $\mathbb{K}^n$.

Quasiperiodic extensions

- A vector in $\mathbb{K}^n$ extends to a unique (n, λ)-qp sequence.
- A subspace of $\mathbb{K}^n$ extends to a unique (n, λ)-qp space.

Example: The $(4, 2)$-quasiperiodic extension of a vector in $\mathbb{K}^4$

$$\begin{pmatrix} 0 & 1 & -1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & 0 & .5 & -.5 & -1 & 0 & 1 & -1 & -2 & 0 & 2 & -2 & -4 & \cdots \end{pmatrix}$$

So why is this interesting?

If we don't fix λ, a vector or subspace in $\mathbb{K}^n$ has a one-parameter family of n-quasiperiodic extensions in $\mathbb{K}^{\mathbb{Z}}$.

Knutson-Lam-Speyer's juggling functions extend to qp-spaces.

Defn: The juggling function π of a quasiperiodic space V

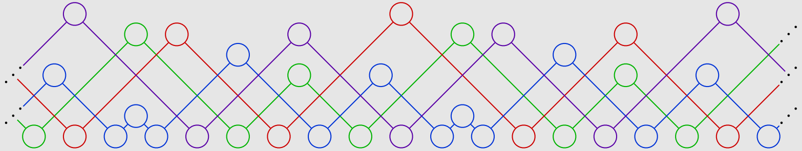
For all $a \in \mathbb{Z}$, define $\pi(a)$ to be the smallest number in $[a, \infty)$ s.t.

$$\dim(V_{[a,\pi(a)]}) = \dim(V_{[a+1,\pi(a)]})$$

Here, $V_{[a,b]}$ is the image of V under the projection $\mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{[a,b]}$.

...Wait, why juggling?

The map π describes a juggling pattern in which, at each moment $a \in \mathbb{Z}$, a juggler throws a ball that is later caught at moment $\pi(a)$.



Properties of juggling functions

Let π be the juggling function of an n-quasiperiodic space V.

- π is a bijection.
- $\pi(a+n) = \pi(a) + n$ for all a.
- $a \leq \pi(a) \leq a + n$ for all a.
- For any a,

$$\dim(V) = \frac{1}{n} \sum_{b=a}^{a+n-1} (\pi(b) - b)$$

This sum is called the number of balls of π.

Juggling functions $\leftrightarrow$ Positroids

A function with these properties is also called a bounded affine permutation, and they are in canonical bijection with positroids.

Like KLS, we may use juggling functions to define a moduli space.

Defn: A quasiperiodic positroid variety

Given an n-periodic juggling function π, let $\mathrm{QpGr}(\pi)$ denote the moduli space of n-quasiperiodic spaces with juggling function π.

This has the structure of an affine $\mathbb{K}$-variety, made explicit below.

Relation between $\mathrm{Gr}(\pi)$ and $\mathrm{QpGr}(\pi)$

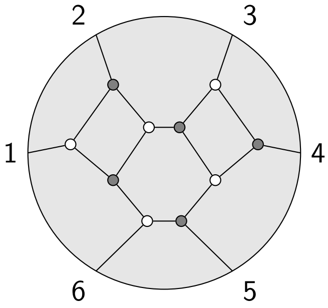
There is an isomorphism of varieties

$$(\mathbb{K}^\times) \times \mathrm{Gr}(\pi) \xrightarrow{\sim} \mathrm{QpGr}(\pi)$$

which sends (λ, V) to the (n, λ)-quasiperiodic extension of V.

Quasiperiodic spaces from plabic graphs

Consider a reduced plabic graph Γ in the disc with a clockwise indexing of its boundary vertices from 1 to n (considered mod n).



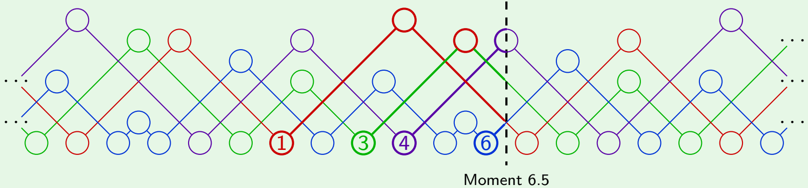
The 'rules of the road' define a juggling function $\pi : \mathbb{Z} \to \mathbb{Z}$ of Γ.

Throwing histories

Given a juggling function π and $a \in \mathbb{Z}$, define

$$T_a := \{b \in (-\infty, a] \mid \pi(b) > a\}$$

This records when the airborne balls after moment a were thrown.



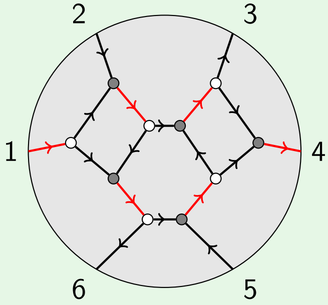
The set $\{T_a \mid a \in \mathbb{Z}\}$ is the reverse Grassman necklace of π.

Lemma (M-Speyer)

A reduced plabic graph Γ with juggling function π admits a unique acyclic perfect orientation whose boundary sources are in T_a.

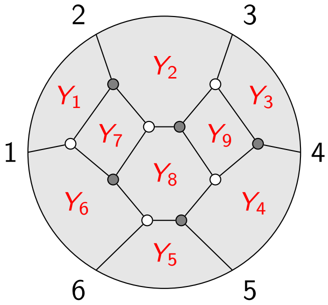
Let us call this the T_a**-orientation** of Γ.

Example: The T_2-orientation



- $T_2 = \{-1, 1, 2\} \equiv \{1, 2, 5\}$.
- The deviant edges of the perfect orientation are in red.
- This orientation is acyclic.
- There are no other perfect orientations with boundary sources T_2.

A face weighting of Γ assigns a weight $Y_f \in \mathbb{K}^\times$ to each face f.



The plan

Use a face weighting of Γ and the n-many T_a-orientations to construct a $\mathbb{Z} \times \mathbb{Z}$-matrix whose kernel is a quasiperiodic space.

Defn: The recurrence matrix of boundary measurements

Given a face weighting Y of Γ, define a $\mathbb{Z} \times \mathbb{Z}$-matrix $\mathrm{C}(Y)$ by

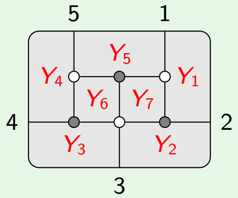
$$\mathrm{C}(Y)_{a,b} := \left\{ \begin{array}{cc} (-1)^{\bullet} \displaystyle\sum_{p:b \to a} (\text{weight left of } p) & \text{if } b \leq a < b+n \\ 0 & \text{otherwise} \end{array} \right\}$$

where the sum is over paths from b to a in the $T_{(a-1)}$-orientation.

Notice the orientation used depends on the endpoint of the path.

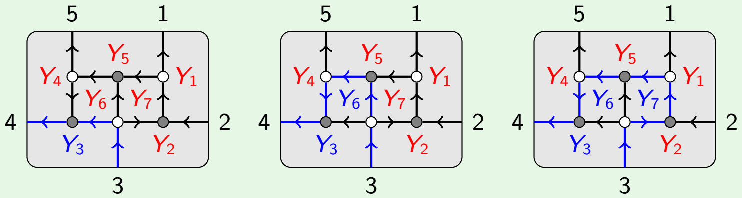
- We use $(-1)^{\bullet}$ to denote a sign we gloss over entirely.
- Exceptions are needed for boundary-adjacent leaves.

Example: Computing the entry $C(Y)_{4,3}$



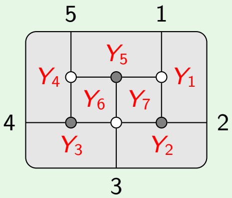
$$\pi(a) = a + 2$$
$$T_a = \{a-1, a\}$$

Consider the three paths from 3 to 4 in the T_3-orientation of this Γ.



$$C(Y)_{4,3} := Y_3 + Y_3 Y_6 + Y_3 Y_6 Y_7$$

Example: The matrix $C(Y)$



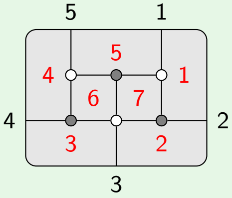
To fit $C(Y)$ on a slide, we rotate it by 45° and delete the 0s:

$$\begin{bmatrix} \cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots & \\ & \cdots & & Y_1(1+Y_7) & & Y_2 & & Y_3(1+Y_6+Y_6Y_7) & & Y_4 & & Y_5(1+Y_6) & & \cdots \\ \cdots & & Y_1Y_5Y_7 & & Y_1Y_2 & & Y_2Y_3Y_6Y_7 & & Y_3Y_4 & & Y_4Y_5Y_6 & & \cdots & \end{bmatrix}$$

To the left and right, the entries repeat 5-periodically.

Example: The matrix $C(Y)$

This may be more clear with explicit face weights.



$$\begin{bmatrix} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 4 & 35 & 8 & 1 & \color{blue}{147} & 4 & 35 & 8 & 1 & \cdots \\ \cdots & 12 & 120 & 35 & 2 & 252 & 12 & 120 & 35 & 2 & \cdots \end{bmatrix}$$

Theorem (DoCampo-M)

The kernel of $\mathrm{C}(Y)$ is (n,λ)-quasiperiodic with juggling function π.

$$\lambda = (-1)^{\bullet} \prod_{f \in F} Y_f$$

Tools in the proof

- An analog of Gessel-Viennot-Lindström's Lemma:

  $$\det(\mathrm{C}(Y)_{[a,b],[c,d]}) = (-1)^{\bullet} \sum_{P} \prod_{p \text{ in } P} (\text{weight to the left of } p)$$

  where the sum runs over vertex-disjoint multipaths from $[c,d]$ to $[a,b]$ in the $T_{(a-1)}$-orientation.
- A determinantal characterization of linear recurrences with quasiperiodic solutions.

This construction ‘extends’ the boundary measurement map.

Theorem (DoCampo-M)

Sending a face weighting to $\ker(C)$ defines an open embedding

$$\beta : (\mathbb{K}^\times)^F \hookrightarrow \mathrm{QpGr}(\pi)$$

which fits into a commutative diagram

$$\begin{array}{ccc} (\mathbb{K}^\times)^E/\mathrm{Gauge} & \xhookrightarrow{\text{Monodromy}} & (\mathbb{K}^\times)^F \\ {\scriptsize\text{Boundary Meas. Map}} \big\downarrow & & \big\downarrow \beta \\ \mathrm{Gr}(\pi) & \xhookrightarrow{(\pm 1)\text{-qp-extension}} & \mathrm{QpGr}(\pi) \end{array}$$

The monodromy map $(\mathbb{K}^E)/\mathrm{Gauge} \hookrightarrow (\mathbb{K}^\times)^F$ weights each face by an alternating product of the weights of adjacent edges.

Tangent: Friezes

Recurrence matrices and friezes

If $\pi(a) = a + k$ for all a and every face has weight 1, then $\mathrm{C}(Y)$ is a tame SL_k-frieze (when rotated 45°).

For other π, we get an analog of friezes with a 'ragged lower edge'.

Friezes

A tame SL_k-frieze is an infinite strip of numbers (offset in a diamond pattern) such that

- the top and bottom rows consist of 1s,
- the determinant of any $k \times k$ diamond is 1, and
- the determinant of any $(k+1) \times (k+1)$ diamond is 0.

Tangent: Twists

Theorem (DoCampo-M)

The kernel of $\mathsf{C}(Y)^\top$ is (n, λ^{-1})-qp with juggling function π.

Every positroid variety has a left twist automorphism.

$$\overleftarrow{\tau} : \mathrm{Gr}(\pi) \to \mathrm{Gr}(\pi)$$

Theorem (DoCampo-M)

The left twist $\overleftarrow{\tau} : \mathrm{Gr}(\pi) \to \mathrm{Gr}(\pi)$ extends to a left twist

$$\overleftarrow{\tau} : \mathrm{QpGr}(\pi) \xrightarrow{\sim} \mathrm{QpGr}(\pi)$$

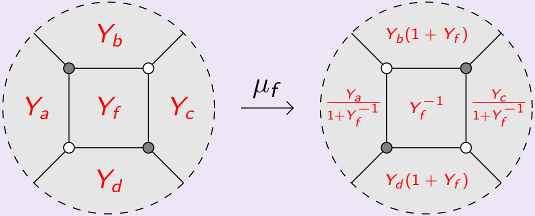
The two quasiperiodic spaces associated to $\mathsf{C}(Y)$ are related by

$$\ker(\mathsf{C}(Y)^\top) = \overleftarrow{\tau}(\ker(\mathsf{C}(Y)))$$

The cluster structure on $\mathrm{QpGr}(\pi)$

Mutation of face weights

Given a plabic graph with a face weighting and a square face f, the mutation at f changes the graph and weights near f as follows.



This gives a rational map $(\mathbb{K}^\times)^F \dashrightarrow (\mathbb{K}^\times)^{F'}$ between face weights.

The operation on graphs is sometimes called urban renewal.

Mutation commutes with the respective β-maps

$$\begin{array}{ccc} (\mathbb{K}^\times)^F & \overset{\mu_f}{\leftarrow\text{-}\text{-}\text{-}\text{-}\rightarrow} & (\mathbb{K}^\times)^{F'} \\ & \searrow \quad \swarrow & \\ & \mathrm{QpGr}(\pi) & \end{array}$$

This extends a mutation relation for monodromy coordinates observed by Postnikov in his original paper (Section 12).

A quick overview of two flavors of cluster variety.

Cluster varieties

A cluster variety X is constructed by gluing together algebraic tori along rational maps defined by cluster mutation.

These are sometimes called 'X-type' cluster varieties/mutation to distinguish from the following variant.

Y-type cluster varieties

The toric charts in a cluster variety can be dualized and glued together along Y-type cluster mutation maps to construct a Y-type cluster variety which is dual to the original.

Y-type cluster mutations were introduced in separate contexts by Fock-Goncharov and Fomin-Zelevinsky.

Theorem (Scott, Postnikov, Leclerc, SSBW, Galashin-Lam)

For all π, $\widehat{\mathrm{Gr}}(\pi)$ is a(n X-type) cluster variety.

- Each plabic graph defines a cluster torus in $\widehat{\mathrm{Gr}}(\pi)$
- Urban renewal gives (X-type) mutation maps between tori.

Not every cluster torus in $\widehat{\mathrm{Gr}}(\pi)$ comes from a plabic graph!

The non-plabic clusters are quite mysterious and a serious roadblock to studying the cluster structure of $\widehat{\mathrm{Gr}}(\pi)$.

Y-type mutation from plabic graphs

Mutation of face weights at a square face f is the Y-type cluster mutation dual to an X-type cluster mutation in $\widehat{\mathrm{Gr}}(\pi)$.

Hence, $\mathrm{QpGr}(\pi)$ has a 'partial' Y-type cluster structure dual to $\widehat{\mathrm{Gr}}(\pi)$, in that it contains the duals of the plabic tori in $\widehat{\mathrm{Gr}}(\pi)$.

Conjecture

This extends to a (complete) Y-type cluster structure on $\mathrm{QpGr}(\pi)$ which makes it into the dual cluster variety to $\widehat{\mathrm{Gr}}(\pi)$.

I.e. the duals to non-plabic tori should also embed into $\mathrm{QpGr}(\pi)$.

Knowing part of a Y-type cluster structure is still enough to define a number of structures on $\mathrm{QpGr}(\pi)$.

Consequences of (partial) Y-cluster structure on $\mathrm{QpGr}(\pi)$

- There is a cluster ensemble map
$$\rho : \widehat{\mathrm{Gr}}(\pi) \longrightarrow \mathrm{QpGr}(\pi)$$
which restricts to a monomial map between each cluster torus and its dual. Here, ρ may be defined using the twist on $\widehat{\mathrm{Gr}}(\pi)$.
- $\beta(\mathbb{R}_+^F) \subset \mathrm{QpGr}(\pi, \mathbb{C})$ does not depend on the choice of Γ, and gives a well-defined totally positive part.
- $\mathrm{QpGr}(\pi)$ has a Poisson structure and a quantization.

The most important consequence of this duality is the Fock-Goncharov conjecture, as reformulated and proven by GHKK.

Parametrizing theta functions

Each tropical point of $\mathrm{QpGr}(\pi)$ defines a theta function on $\widehat{\mathrm{Gr}}(\pi)$. The theta functions collectively form a strongly positive basis for the coordinate ring of $\widehat{\mathrm{Gr}}(\pi)$ containing the cluster monomials.

Application to representation theory

Since base affine space is a positroid variety, the theta basis in this case gives a distinguished basis of each simple SL_n-representation.

Big question

What are the tropical quasiperiodic spaces and what are the corresponding theta functions on positroid varieties?