From dimers in the disc to cluster categories

arXiv:1912.12475
and work in progress with İ. Çanakçı and A. King

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Dimers in Combinatorics and Cluster Algebras
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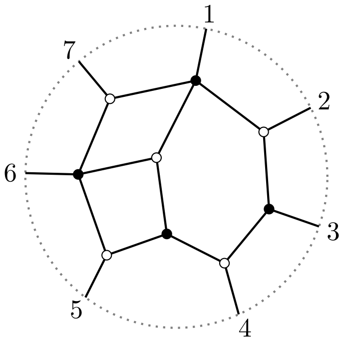
Dimer models



We begin with a dimer model D (a bipartite plabic graph) in the disc.

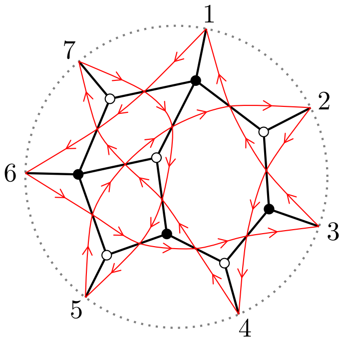
The dimer model should be consistent, meaning that this strands obtained by following the rules of the road form a Postnikov diagram.

The only non-automatic condition here is that strands which cross twice should be oppositely oriented between these crossings—this also rules out closed strands in the interior.

The dimer model has a chirality $k = (k_\bullet, k_\circ)$ with $k_\bullet + k_\circ = n$, the number of boundary marked points, and a permutation σ_D of these points.

This data determines a number of further geometric and algebraic objects, which we will explore.

Dimer models



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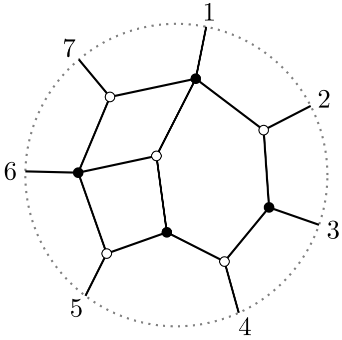
The quiver

The dimer model D cuts the disk into regions, and thus determines a quiver Q_D with

(Q_0) vertices corresponding to the regions

(Q_1) arrows corresponding to edges, oriented with the black vertex on the left.



The vertices and arrows on the boundary—marked in blue and called *frozen*—sometimes play a different role to the others.

In the Postnikov diagram, the vertices correspond to alternating regions, and the arrows to crossings, with their natural orientation.

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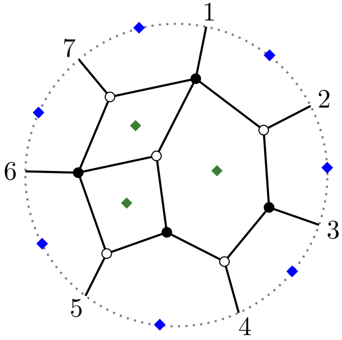

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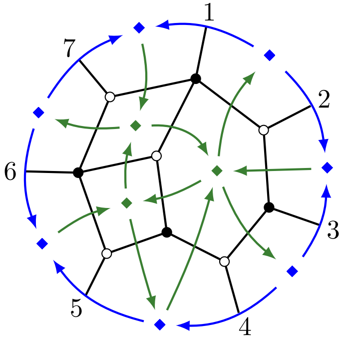

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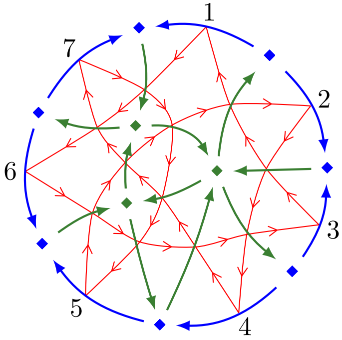

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A cluster algebra

The permutation σ_D is a Grassmann permutation, and hence determines a particular open *positroid* subvariety $\Pi(\sigma_D) \subseteq \mathrm{Gr}^n_{k_\bullet}$ of the Grassmannian of $k_\bullet$-dimensional subspaces of $\mathbb{C}^n$ [Postnikov].

It also determines a cluster algebra $\mathscr{A}_D$, with invertible frozen variables, via the quiver Q_D.

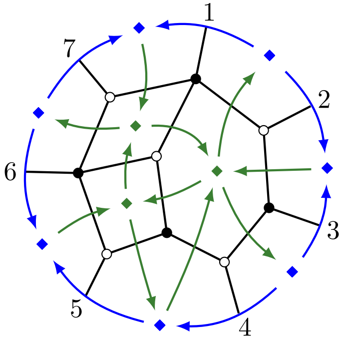
Theorem (Serhiyenko–Sherman-Bennett–Williams, Galashin–Lam)

There is an isomorphism $\mathscr{A}_D \xrightarrow{\sim} \mathbb{C}[\Pi(\sigma_D)]$, mapping the initial cluster variables to restrictions of Plücker coordinates.

For $\sigma_D \colon i \mapsto i + k_\circ \mod n$ (the *uniform permutation*), the variety $\Pi(\sigma_D)$ is dense in $\mathrm{Gr}^n_{k_\bullet}$, and the cluster algebra with non-invertible frozen variables attached to Q_D is isomorphic to the homogeneous coordinate ring $\mathbb{C}[\widehat{\mathrm{Gr}}^n_{k_\bullet}]$. [Scott]

In this case, Jensen–King–Su have categorified the cluster algebra—our aim is to extend this to more general positroid varieties.

A non-commutative algebra



The dimer model D gives Q_D a determined set of $\bullet$-cycles and $\circ$-cycles (bounding faces).

Thus, letting $Z = \mathbb{C}[[t]]$, we can consider matrix factorisations on Q_D: representations with free Z-modules at each vertex, all having the same fixed rank, and in which each $\bullet$- and $\circ$-cycle acts by t. (When the rank is 1, these are given by perfect matchings.)

When D is connected as a graph (equivalently $|Q_D|$ is a topological disc) these are precisely the A_D-modules free over Z, where A_D is the $\mathbb{C}$-algebra determined by the following relations on Q:

Each non-boundary (green) arrow a can be completed to either a $\bullet$-cycle or a $\circ$-cycle by unique paths $p_a^\bullet$ and p_a°; we impose each relation $p_a^\bullet = p_a^\circ$.

This is an example of a *frozen Jacobian algebra*, for the potential $W = \sum(\bullet\text{-cycles}) - \sum(\circ\text{-cycles})$.

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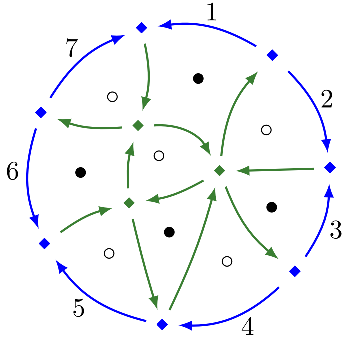

The dimer model D gives Q_D a determined set of •-cycles and ○-cycles (bounding faces).

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The boundary algebra

Let $e = e^2 \in A_D$ be the sum of vertex idempotents at boundary vertices, and write $B_D = eA_De$ for the *boundary algebra*.

Theorem (Jensen–King–Su, Baur–King–Marsh)

When σ_D is the uniform permutation, the category

$$\mathrm{GP}(B_D) = \{X \in \operatorname{mod} B_D : \operatorname{Ext}^{>0}_{B_D}(X, B_D) = 0\}$$

categorifies the cluster algebra $\mathscr{A}_D$.

Note: this presentation is historically backwards. In practice, Jensen–King–Su proved the above theorem for an explicitly defined 'circle algebra' C_k, depending only on the chirality, which Baur–King–Marsh (slightly) later showed is isomorphic to B_D whenever σ_D is the uniform permutation.

With hindsight, we can try to repeat this trick, this time using the boundary algebra description of B_D as the (now more general) definition.

Categorification

Theorem

Let D be a connected consistent dimer model in the disc, with dimer algebra $A = A_D$ and boundary algebra $B = B_D$. Then

(1) *B is Iwanaga–Gorenstein of Gorenstein dimension $\leqslant 3$; that is, B is Noetherian and $\operatorname{injdim}{}_B B, \operatorname{injdim} B_B \leqslant 3$. In particular $\mathrm{GP}(B)$ is a **Frobenius** category.*

(2) *The stable category $\underline{\mathrm{GP}}(B) = \mathrm{GP}(B)/\operatorname{proj} B$ is a **2-Calabi–Yau** triangulated category.*

(3) *$A = \operatorname{End}_B(eA)^{\mathrm{op}}$ and $eA \in \mathrm{GP}(B)$ is **cluster-tilting**, that is*

$$\operatorname{add}(eA) = \{X \in \mathrm{GP}(B) : \operatorname{Ext}^1_B(X, eA) = 0\}.$$

This theorem follows from the following facts about the pair (A, e):

(1) A is Noetherian,

(2) A/AeA is finite-dimensional, and

(3) A is internally bimodule 3-Calabi–Yau with respect to e.

Internally Calabi–Yau algebras

The definition of A being internally bimodule 3-Calabi–Yau algebra is technical, and we omit it, but it implies that $\mathrm{gl.dim}\, A \leqslant 3$ and that

$$\mathrm{Ext}^i_A(X, Y) = \mathrm{Ext}^{3-i}_A(Y, X)^*$$

for $X, Y \in \mathrm{mod}\, A$ with $eY = 0$.

The result is analogous to Broomhead's theorem that a consistent dimer model on the torus is bimodule 3-Calabi–Yau in the usual (no boundary) sense.

The proof that A_D has this property uses that it is a frozen Jacobian algebra—hence the restriction to D connected—and the thinness property ($e_i A e_j \cong Z$) obtained with Çanakçı and King (which also implies the required Noetherianity / finite-dimensionality).

Warning

When σ_D is the uniform permutation, $\mathrm{GP}(B_D) = \mathrm{CM}(B_D)$, i.e. it consists of those B_D-modules free and finitely generated over Z. In general, $\mathrm{GP}(B_D)$ is a proper full subcategory of $\mathrm{CM}(B_D)$.

Relationship to the JKS category

Proposition (Çanakçı–King–P)

For D of chirality k, there is a fully faithful functor $\mathrm{CM}(B_D) \to \mathrm{CM}(C_k)$, recalling that C_k is JKS's circle algebra for the uniform permutation.

The cluster-tilting object $eA_D \in \mathrm{GP}(B_D)$ is sent to to a direct sum of rank 1 (Plücker) modules M_J, for the labels J attached to alternating regions of the Postnikov diagram of D by labelling regions using the sources of strands.

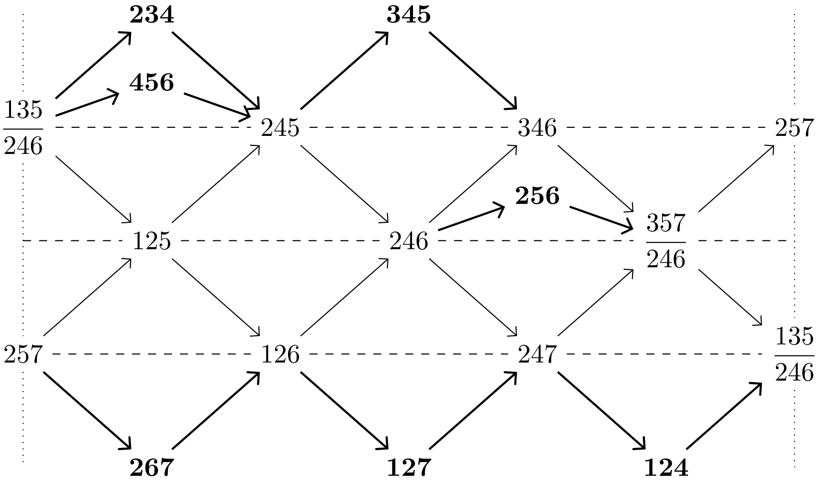
The functor in the first statement arises from a natural map $C_k \to B_D$ (not surjective in general).

The second statement follows from the fact that restricting the projective A_D-module at vertex i to B_D and then to C_k produces the rank 1 module M_J, for J is obtained by the given rule.

What of target labelling? By duality there is another Frobenius category $\mathrm{GI}(B_D) \subset \mathrm{CM}(B_D)$ containing the cluster tilting object $\mathrm{Hom}_Z(A_De, Z)$, which restricts to the direct sum of the rank 1 modules coming from this labelling rule.

Example

In the running example, $\mathrm{GP}(B_D)$ is (inside $\mathrm{CM}(C_k)$) as shown:



Application to twists with İ. Çanakçı + A. King

Fix D consistent and connected, and write $A = A_D$, $B = B_D$, $F = \mathrm{Hom}_B(eA, -)$, and $G = \mathrm{Ext}^1_B(eA, -)$.

To $X \in \mathrm{CM}(B)$ we attach the Laurent polynomial

$$\Phi_X = x^{[FX]} \sum_{E \leqslant GX} x^{-[E]}$$

where $[M]$ is the class of M in $\mathrm{K}_0(A)$ (written in the basis of indecomposable projectives), and infinite sums are computed using Euler characteristics of quiver Grassmannians. This is the CC-formula, computing cluster monomials from rigid objects.

If $PX \to X$ is a projective cover with kernel ΩX, there is an exact sequence

$$FPX \to FX \to G\Omega X \to 0,$$

and we write $F'X$ for the image of the left-hand map. Then

$$\Phi_{\Omega X} = x^{[FPX]} \sum_{F'X \leqslant N \leqslant FX} x^{-[N]}$$

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Proposition (Çanakçı–King–P)

Consider $X = M_J$ (i.e. $X \in \mathrm{CM}(B)$ restricts to this C_k-module). Then $\{F'M_J \leqslant N \leqslant FM_J\}$ is the set of perfect matching modules N_μ for matchings μ of D with $\partial\mu = J$.

This result uses the categorification theorem, specifically that $FB = eA$ and $GB = 0$.

It allows us to compare the CC-formula to the Marsh–Scott formula

$$\mathrm{MS}(J) = x^{-\,\mathrm{wt}(D)} \sum_{\mu : \partial\mu = J} x^{\mathrm{wt}(\mu)}$$

where $\mathrm{wt}(D)$ and $\mathrm{wt}(\mu)$ are combinatorially defined weights in $\mathrm{K}_0(A)$.

Application to twists with İ. Çanakçı + A. King

$$\Phi_{\Omega X} = x^{[FPX]} \sum_{F'X \leqslant N \leqslant FX} x^{-[N]} \qquad \mathrm{MS}(J) = x^{-\,\mathrm{wt}(D)} \sum_{\mu:\partial\mu = J} x^{\mathrm{wt}(\mu)}$$

Theorem (Çanakçı–King–P)

For $M_J \in \mathrm{CM}(B)$, there is a canonical choice of projective cover $\mathbf{P}M_J \to M_J$, inducing a canonical syzygy $\mathbf{\Omega}M_J$, for which

$$\Phi_{\mathbf{\Omega}M_J} = \mathrm{MS}(J).$$

Under a suitable specialisation of the x_i to Plücker coordinates, taking Φ_{M_J} to the Plücker coordinate Δ_J, the Laurent polynomial $\mathrm{MS}(J)$ evaluates to the Marsh–Scott twist of Δ_J. That is, $\mathbf{\Omega}$ categorifies this twist.

The theorem is proved by computing a projective resolution of each perfect matching A-module N_μ, from which it follows that $[F\mathbf{P}M_J] - [N_\mu] = \mathrm{wt}(\mu) - \mathrm{wt}(D)$ for each perfect matching μ with $\partial\mu = J$.

Thanks for listening!