Reciprocity, Quasirandomness, and Resonance

James Propp, UMass Lowell

presented as part of
DIMERS 2020
August 5, 2020

Slides at http://jamespropp.org/dimers20.pdf
Video at ...

Acknowledgment

Versions of some of these questions have been discussed on three electronic forums I run (DOMINO, ROBBINS, and DAC) and with several generations of combinatorialists (especially from REACH).

If I forget to cite relevant work through ignorance or forgetfulness, please speak up and educate/remind me!

Stanley reciprocity

Let $T(m,n)$ denote the number of ways to tile an m-by-n rectangle with dominos.

For each $m \geq 0$, we can extend to all n in $\mathbb{Z}$.

Stanley: For $m, n \geq 0$, $T(m, -2-n) = \epsilon_{m,n} T(m,n)$, where $\epsilon_{m,n}$ is -1 if m is congruent to 2 (mod 4) and n is odd, and is $+1$ otherwise.

Example: $m = 2$; reciprocity for the two-sided Fibonacci sequence

$$\ldots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots$$

I offer a combinatorial explanation in my article A reciprocity theorem for domino tilings.

Figurate numbers

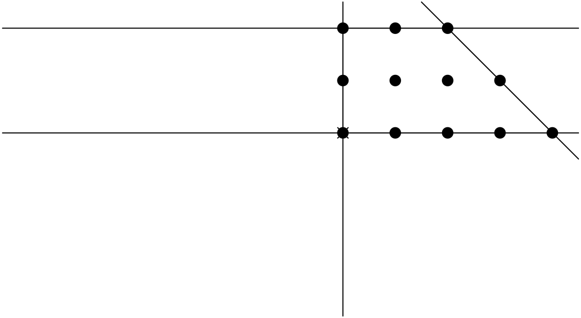
The nth triangle number $n(n+1)/2$ is also the $(-1-n)$th triangle number.

The nth square number n^2 is also the $(-n)$th square number.

The nth pentagonal number of the first kind $n(3n-1)/2$ is also the $(-n)$th pentagonal number of the second kind $n(3n+1)/2$.

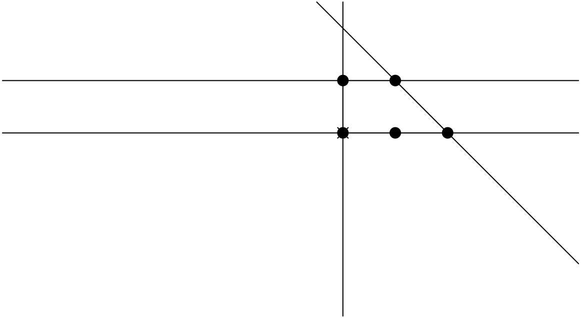
Ehrhart theory

My favorite way of understanding this comes from Ehrhart theory.



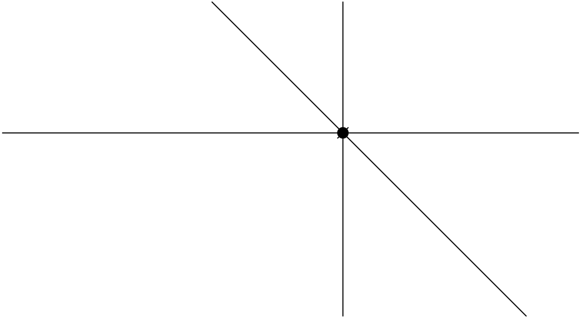
Ehrhart theory

My favorite way of understanding this comes from Ehrhart theory.



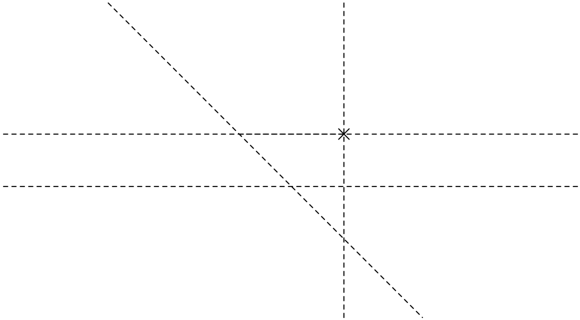
Ehrhart theory

My favorite way of understanding this comes from Ehrhart theory.



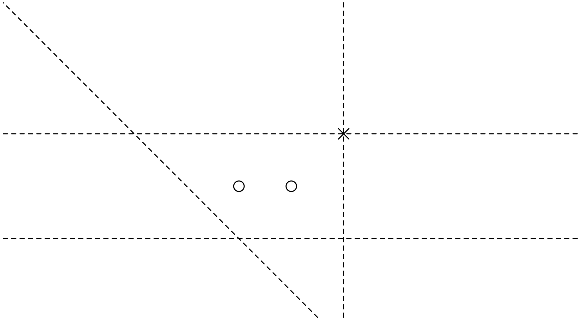
Ehrhart theory

My favorite way of understanding this comes from Ehrhart theory.



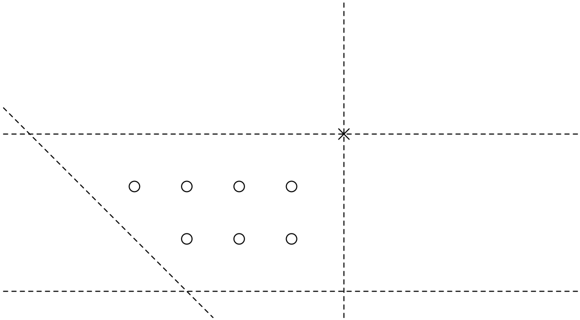
Ehrhart theory

My favorite way of understanding this comes from Ehrhart theory.



Ehrhart theory

My favorite way of understanding this comes from Ehrhart theory.



Ehrhart theory

(Note that for negative dilation factors, the game is slightly different.)

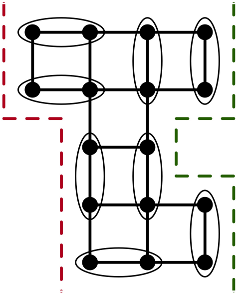
Can we get something like this for dimer-state enumerations?

Speyer's reciprocity formula

$$F(G, n, P, Q)(x) = F(G, -n, \overline{P}, \overline{Q})(-x)$$

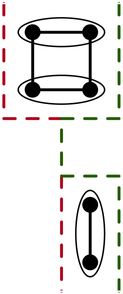
To find out what the formula means and why it's true, see Speyer's writeup.

Anzalone et al.



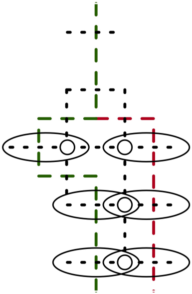
23 matchings

Anzalone et al.



2 matchings

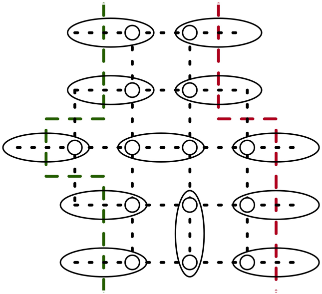
Anzalone et al.



0 matchings

(all edges joining a vertex to nowhere must be used)

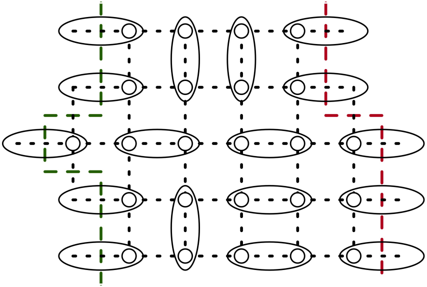
Anzalone et al.



-1 signed matchings

(all edges joining a vertex to nowhere must be used;
dashed vertical edges have weight -1)

Anzalone et al.



-8 signed matchings

(all edges joining a vertex to nowhere must be used;
dashed vertical edges have weight -1)

Anzalone et al.

$$\ldots, -8, -1, 0, 2, 23, \ldots$$

satisfies

$$f(n+3) - 18f(n+2) + 87f(n+1) - 126f(n)$$
$$+ 87f(n-1) - 18f(n-2) + f(n-3) = 0$$

What next?

All this is essentially 1-dimensional (even the $T(m,n)$ result, because we hold m fixed and let n grow).

What about genuinely 2-dimensional dimer problems?

The number of domino tilings of an Aztec diamond of order n is $2^{n(n+1)/2}$.

The number of diabolo tilings of a fortress of order n is 5^{n^2} or twice that, depending.

Can we find an Ehrhart-esque explanation of the symmetries in n?

Can we find two dimer enumeration problems that are mutually reciprocal in the manner of the two kinds of pentagonal numbers, with a geometric picture that explains the reciprocity?

Tools?

One tool is the octahedron recurrence, which can often be used to extend a discrete function beyond its initial, combinatorially meaningful domain (unless one encounters 0/0).

From my article Enumeration of tilings: “In the case of the 2-by-2n grid, and for other 1-dimensional matching problems, the beginnings of a satisfactory explanation for reciprocity can be found in [139] and [1]. However, for 2-dimensional tiling problems, nothing has been done. Since shuffling can be done in reverse, one might guess that the octahedron recurrence would provide the scaffolding for an explanation, via an extension of Kuo condensation to signed graphs like the ones considered by [1] (featuring various sorts of negative vertices and edges). Indeed, the rich octahedral symmetries of solutions to the octahedron recurrence call out for such a point of view, in which there would not be such a stark difference between ‘space’ and ‘time’, let alone between the positive and negative time-directions.”

Tools?

There's lots of software for enumeration of perfect matchings of planar graphs via determinants/Pfaffians.

Most 2-D dimer enumeration problems don't have closed-form enumerations, so it's not clear how to replace n by a negative integer.

Potential solution: Look for patterns in enumeration mod n for various n. (Cf. Cohn's work on 2-adic properties of enumeration of domino tilings.)

Payoffs?

Might some sort of theory of perfect matchings of virtual graphs unify disparate cases in complicated proofs?

Big if true

Let $H(a, b, c)$ be the number of lozenge tilings of an a, b, c, a, b, c hexagon, defined for $a, b, c \geq 0$.

If I recall correctly, when you try to extend this function by the trick of holding two variables fixed and freeing the third, and assuming $H(a, b, c)$ is symmetric in a, b, and c, you don't get a single-valued extension to $\mathbb{Z}^3$.

References

Anzalone, Baldwin, Bronshtein, and Petersen, A reciprocity theorem for monomer-dimer coverings

Cohn, 2-adic behavior of numbers of domino tilings

Propp, Negative numbers in combinatorics (slides)

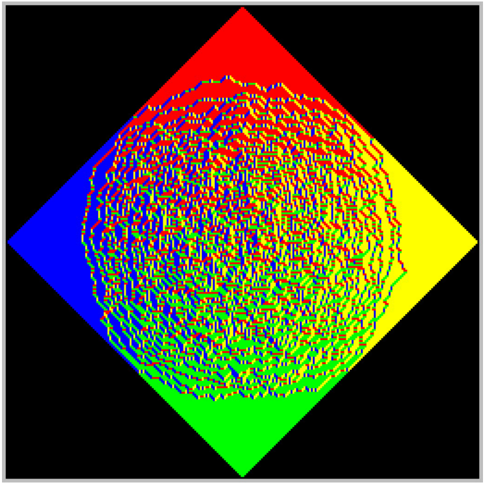
Propp, A reciprocity theorem for domino tilings

Speyer, A reciprocity sequence for perfect matchings of linearly growing graphs (unpublished)

Stanley, On dimer coverings of rectangles of fixed width

Randomness and quasirandomness

The arctic circle theorem for domino tilings of Aztec diamonds says that most tilings of large Aztec diamonds have an approximately circular temperate zone:



What's bothering me

More precisely: If you generate a tiling uniformly at random (e.g. using domino shuffling with fair coin flips), then with probability going to 1 as n gets large, the rescaled temperate zone and the disk inscribed in the rescaled Aztec diamond have Hausdorff distance going to 0 as n gets large.

But how can you find such a tiling NON-randomly?

Idea #1

"Use the bits of π."

(It probably works, but we're at a standoff: you can't prove that it works, and I can't prove that it doesn't.)

Idea #2

"Use the circularity desideratum as a selection criterion."

This is cheating!

I'm not sure how cheating should be defined, but I can recognize it when I see it.

Analogy: IDLA

The discrete stochastic process called Internal Diffusion-Limited Aggregation (IDLA) grows random blobs in the square grid that converge to circularity in the infinite-size limit.

Can we derandomize IDLA so that it still converges to circularity as n gets large?

Yes: with rotor-routing (see Holroyd et al.).

In fact, Levine and Peres showed that as long as each site ejects the walkers in each of the four possible directions close to $1/4$ of the time, the blob built by the walkers will converge to circularity.

Randomness is only relevant insofar as it guarantees the equidistribution property with probability 1.

Idea #3

Quasirandom domino-shuffling?

Doesn't seem to work.

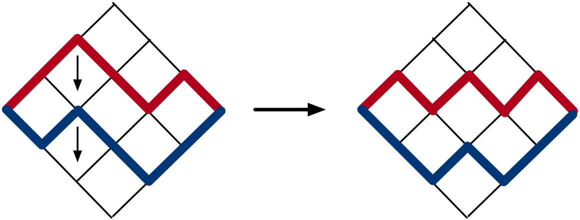
New idea

For domino tilings of Aztec diamonds OR lozenge tilings of hexagons:

Do quasirandom coupling from the past, using the fact that in both cases, height functions give a partial order on tilings.

For simplicity, I'll focus on a height-function in 1+1 dimensions (instead of 2+1 dimensions) and I'll ignore the difference between coupling from the past and coupling into the future.

Coupled swaps



1 1 0 0 1 0	1 **0** **1** 0 1 0
0 1 0 0 1 1	0 **0** **1** 0 1 1

Do the upper path and lower path coalesce? If so, what do they coalesce to?

Try it with coins

1 1 0 0

0 0 1 1

Try it with coins

1 1 0 0

0 0 1 1

→

1 0 0 1

0 0 1 1

Try it with coins

1 1 0 0

0 0 1 1

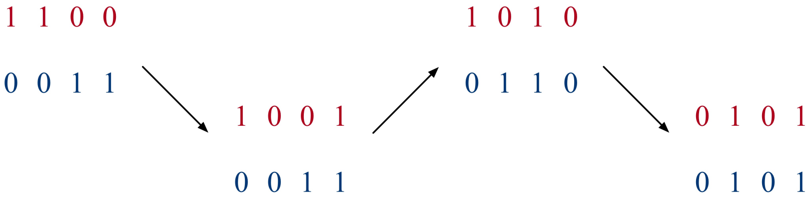
1 0 0 1

0 0 1 1

1 0 1 0

0 1 1 0

Try it with coins



Does this always get the coins to intermingle evenly?

No; try it with unequal numbers of 0s and 1s and you'll see the problem.

Challenge: Find a protocol for coupling mutations that intermingles a 0's and b 1's (i.e., generates order ideals in $[a] \times [b]$ that are close to the average) for arbitrary a, b and then try to adapt it to lozenge tilings of an a, b, c, a, b, c hexagon (i.e., order ideals in $[a] \times [b] \times [c]$).

References

Holroyd, Levine, Meszaros, Peres, Propp, and Wilson, Chip-firing and rotor-routing on directed graphs

Levine and Peres, Spherical asymptotics for the rotor-router model in Z^d

Propp, Lattice structure for orientations of graphs (new 2020 version)

Propp and Wilson, Exact sampling with coupled Markov chains applications to statistical mechanics

Pseudoperiodicity conjecture (?) for ASMs

For this talk, an alternating sign matrix will not be a matrix or a fully packed collection of loops but an order ideal in a particular ranked poset P (as explained for instance in Striker's article).

Let $J(P)$ be the set of order ideals of P.

For $x \in P$ and $I \in J(P)$, let

$$\tau_x(I) = \begin{cases} I \triangle \{x\} & \text{if } I \triangle \{x\} \in J(P), \\ I & \text{otherwise} \end{cases}$$

(τ_x is called "toggling at x").

Pseudoperiodicity conjecture (?) for ASMs

Let Row : $J(P) \to J(P)$ be the composition of all the toggle operators τ_x in order of rank (this can be shown to be well-defined).

(Here we ignore the difference between Row and Row^{-1} since we're only concerned with orbit structure.)

Let Gyr : $J(P) \to J(P)$ be the composition of all the toggle operators τ_x in order of rank with all even-rank toggles preceding all odd-rank toggles (this too can be shown to be well-defined).

Theorem (special case of theorem of Striker and Williams): Row and Gyr have the same orbit structure.

Pseudoperiodicity conjecture (?) for ASMs

Conjecture: When n is large, almost every ASM of order n belongs to an orbit whose size is divisible by $2n$, and $2n$ is the largest number with this property.

That is, $2n$ is the "greatest asymptotically-common divisor" of the orbit sizes.

[Has anyone proved this?]

Why you should believe it

1) There's an equivariant surjection (see Wieland's article) that maps gyration of ASMs to rotation of link patterns (noncrossing matchings of $2n$ points on a circle) such that for every ASM A,

$$\#(\mathrm{Orbit}_{\mathrm{Gyration}}((A)))$$

is a multiple of

$$\#(\mathrm{Orbit}_{\mathrm{Rotation}}(\mathrm{LinkPattern}(A))).$$

2) Most link patterns have no symmetries.

3) Most ASMs map to link patterns that have no symmetries.

Broadening the game

Since the states of a plane bipartite dimer model form a distributive lattice, and hence can be seen as order ideals in a poset, rowmotion gives rise to a large class of potentially interesting actions.

Moreover, these rowmotion actions are orbit-equivalent to pictorially natural gyration actions on dimer states.

Lozenge tilings of hexagons

Gyration means "Do face moves in the hexagons centered at red vertices, then the hexagons centered at green vertices, then the hexagons centered at blue vertices" (with respect to the tricoloring of the triangular grid).

"Known": rowmotion for lozenge tilings of the a, b, c, a, b, c hexagon has pseudoperiod $a + b + c - 1$.

Proved when $\min(a, b, c)$ is 2 (see Cameron and Fon-Der-Flaass), when $a + b + c - 1$ is prime (see Patrias and Pechenik), and when $c >> a, b$ (see Dilks-Pechenik-Striker).

Why you should believe it: K-promotion

Rowmotion/gyration for dimer models in a square grid

Gyration means "Do face moves in the squares centered at black vertices, then the squares centered at white vertices" (with respect to the bicoloring of the square grid).

Does this action exhibit nontrivial pseudoperiodicity?

References

Cameron and Fon-Der-Flaass, Orbits of antichains revisited

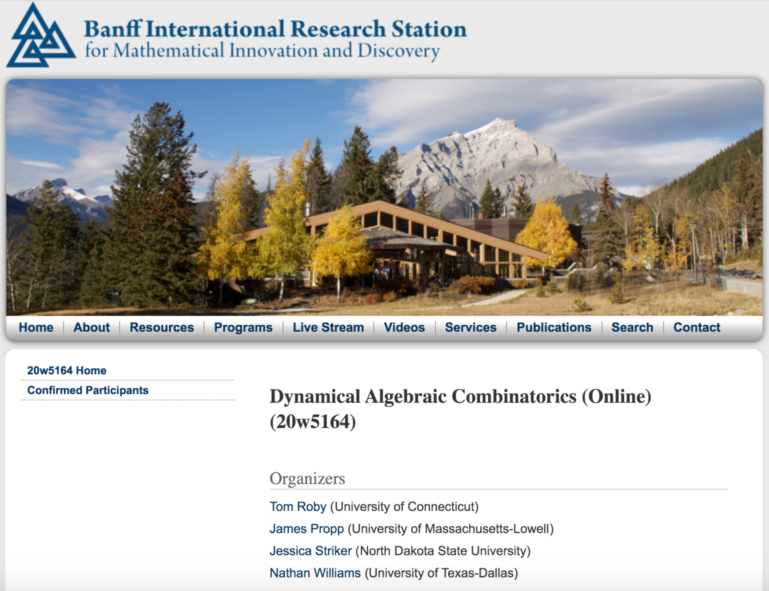
Dilks, Pechenik, and Striker, Resonance in orbits of plane partitions and increasing tableaux

Patrias and Pechenik, Proof of the Cameron and Fon-Der-Flaass periodicity conjecture

Striker, A unifying poset perspective on alternating sign matrices, plane partitions, Catalan objects, tournaments, and tableaux

Wieland, A large dihedral symmetry of the set of alternating sign matrices

If the last two problems interest you



If the last two problems interest you

There are two upcoming conferences in dynamical algebraic combinatorics:

- Virtual conference open to all, Oct. 19,21,23,26,28,30 (tentatively 11am-2pm EST)
- In-person workshop at BIRS, date TBA

Email me if you're interested.

Thank you

Thanks to the people who organized this conference and to everyone who listened to this talk, and special thanks to anyone who did both!

Slides for this talk at http://jamespropp.org/dimers20.pdf