Dimers and embeddings

Marianna Russkikh

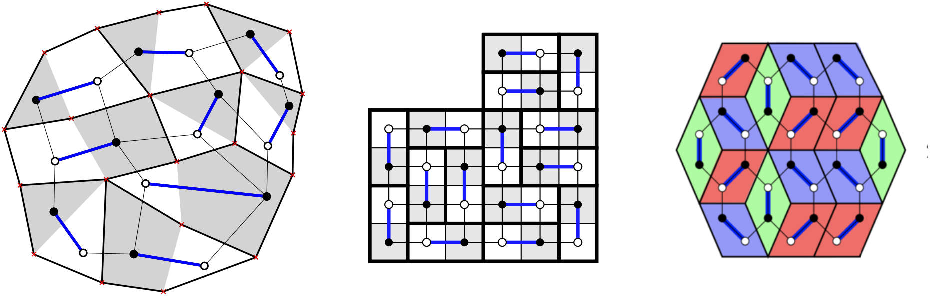
MIT

Based on:

[KLRR] "Dimers and circle patterns"
joint with R. Kenyon, W. Lam, S. Ramassamy. (arXiv:1810.05616)

[CLR] "Dimer model and holomorphic functions on t-embeddings"
joint with D. Chelkak, B. Laslier. (arXiv:2001.11871)

Dimer model

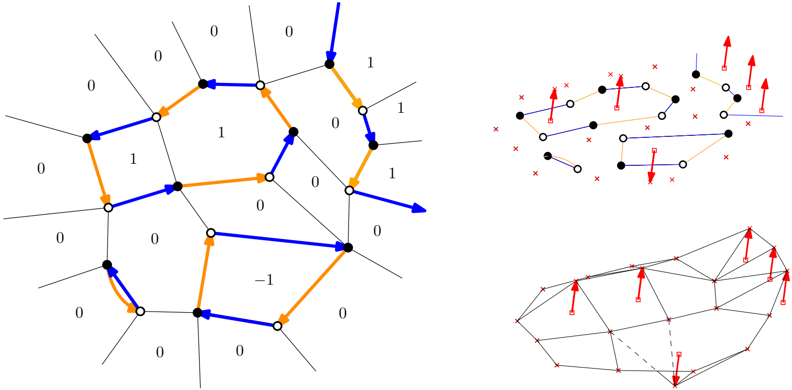


A dimer cover of a planar bipartite graph is a set of edges with the property: every vertex is contained in exactly one edge of the set.

(On the square lattice / honeycomb lattice it can be viewed as a tiling of a domain on the dual lattice by dominos / lozenges.)

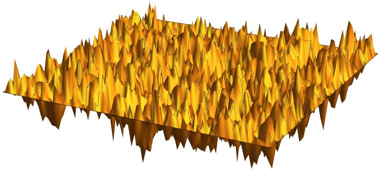
Height function

Defined on $\mathcal{G}^*$, fixed reference configuration, random configuration



Note that $(h - \mathbb{E}h)$ doesn't depend on the reference configuration.

Gaussian Free Field



GFF with zero boundary conditions on a domain $\Omega \subset \mathbb{C}$ is a conformally invariant random generalized function:

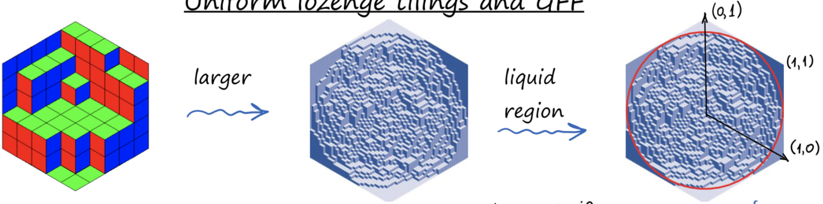
$$\mathrm{GFF}(z) = \sum_k \xi_k \frac{\phi_k(z)}{\sqrt{\lambda_k}},$$

[1d analog: Brownian Bridge]

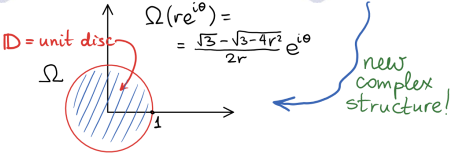
where ϕ_k are eigenfunctions of $-\Delta$ on Ω with zero boundary conditions, λ_k is the corresp. eigenvalue, and ξ_k are i.i.d. standard Gaussians.
The GFF is not a random function, but a random distribution.

GFF is a Gaussian process on Ω with Green's function of the Laplacian as the covariance kernel.

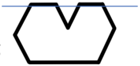
Uniform lozenge tilings and GFF



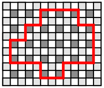
Theorem As mesh goes to zero,
Fluctuations of height ⇒
Gaussian Free Field on $\mathbb{D}$ with
zero boundary conditions.



- *[Kenyon '01+] conjectured for general lattices/domains, proved for lozenge tilings without facets in the limit shape.*
- *[Petrov '12], [Bufetov–Gorin '16–17]: certain polygons*



[Kenyon'08], [Berestycki–Laslier–Ray' 16]: lozenge tilings
[Kenyon'00], [R.'16-18]: domino tilings
(open question: domains composed of 2 × 2 blocks on $\mathbb{Z}^2$)

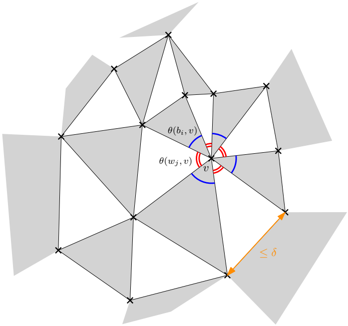


$\hbar = h - \mathbb{E}h$

Ambitious goal [Chelkak, Laslier, R.]:
Given a big weighted bipartite planar graph to embed it so that

$$\hbar^\delta \to \mathrm{GFF}$$

Q: In which metric?



$$(\mathcal{G}, K) \to (\mathcal{T}(\mathcal{G}^*), K_{\mathcal{T}}), \quad K \underset{\text{gauge}}{\sim} K_{\mathcal{T}}$$

t-embedding
or
circle pattern embedding

Results

Theorem (Kenyon, Lam, Ramassamy, R.)

t-embeddings exist at least in the following cases:

- *If $\mathcal{G}^\delta$ is a bipartite finite graph with outer face of degree 4.*
- *If $\mathcal{G}^\delta$ is a biperiodic bipartite graph.*

Theorem (Chelkak, Laslier, R.)

Assume $\mathcal{G}^\delta$ are perfectly t-embedded.

a) *Technical assumptions on faces*

b) *The origami map is small in the bulk*

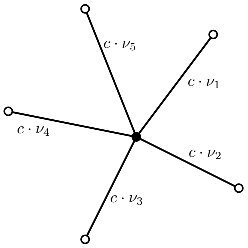
$\Rightarrow$ *convergence to $\pi^{-1/2}\,\mathsf{GFF}_{\mathbb{D}}$.*

Theorem (Affolter; Kenyon, Lam, Ramassamy, R.)

Circle pattern embeddings / t-embeddings of $\mathcal{G}^*$ are preserved under elementary transformations of $\mathcal{G}$.

Application: Miquel dynamics.

Weighted dimers and gauge equivalence



Weight function $\nu : E(\mathcal{G}) \to \mathbb{R}_{>0}$

Probability measure on dimer covers:

$$\mu(m) = \frac{1}{Z} \prod_{e \in m} \nu(e)$$

Definition

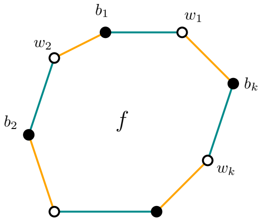
Two weight functions ν_1, ν_2 are said to be *gauge equivalent* if there are two functions $F : B \to \mathbb{R}$ and $G : W \to \mathbb{R}$ such that for any edge bw, $\nu_1(bw) = F(b)G(w)\nu_2(bw)$.

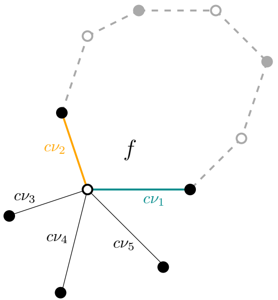
Gauge equivalent weights define the same probability measure μ.

Face weights

For a planar bipartite graph, two weight functions are gauge equivalent if and only if their face weights are equal, where the face weight of a face with vertices $w_1, b_1, \dots, w_k, b_k$ is

$$X_f := \frac{\nu(w_1 b_1) \dots \nu(w_k b_k)}{\nu(b_1 w_2) \dots \nu(b_k w_1)}.$$

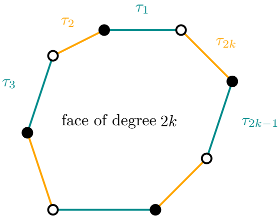




Kasteleyn matrix

Complex Kasteleyn signs:
$\tau_i \in \mathbb{C}$, $|\tau_i| = 1$,

$$\frac{\tau_1}{\tau_2} \cdot \frac{\tau_3}{\tau_4} \cdot \ldots \cdot \frac{\tau_{2k-1}}{\tau_{2k}} = (-1)^{(k+1)}$$

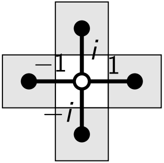


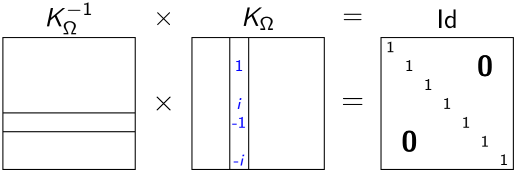
A **(Percus–)Kasteleyn matrix** K is a weighted, **signed** adjacency matrix whose rows index the white vertices and columns index the black vertices: $K(w,b) = \tau_{wb} \cdot \nu(wb)$.

- [Percus'69, Kasteleyn'61]: $Z = |\det K| = \sum_{m \in M} \nu(m)$
- The local statistics for the measure μ on dimer configurations can be computed using **the inverse Kasteleyn matrix**.

Kasteleyn matrix as a discrete Cauchy–Riemann operator

Kasteleyn $\mathbb{C}$ signs proposed by **Kenyon** for the uniform dimer model on $\mathbb{Z}^2$ [flat case]:



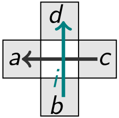


Relation for 4 values of K_Ω^{-1}:

$$1 \cdot K_\Omega^{-1}(v+1, v') - 1 \cdot K_\Omega^{-1}(v-1, v') +$$
$$i \cdot K_\Omega^{-1}(v+i, v') - i \cdot K_\Omega^{-1}(v-i, v') = \delta_{\{v=v'\}}$$

Discrete Cauchy-Riemann:

$$F(c) - F(a) = -i \cdot (F(d) - F(b))$$



Kasteleyn matrix as a discrete Cauchy–Riemann operator

What about non-flat case / general weights / other grids?

A function $F^\bullet : B \to \mathbb{C}$ is discrete holomorphic at $w \in W$ if

$$[\bar{\partial}F^\bullet](w) := \sum_{b\sim w} F^\bullet(b) \cdot K(w,b) = [F^\bullet K](w) = 0.$$

For a fixed $w_0 \in W$ the function $K^{-1}(\,\cdot\,, w_0)$ is a discrete holomorphic function with a simple pole at w_0.

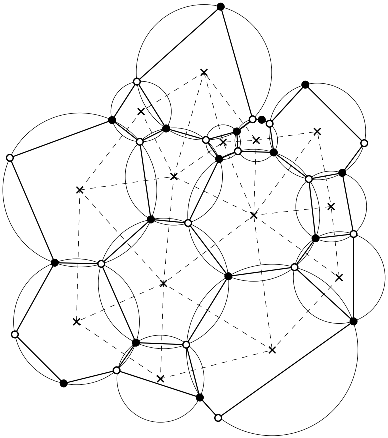
Q: How do discrete holomorphic functions correspond to their continuous counterparts? [gauge + Kasteleyn signs + embedding]

(+) **[flat]** uniform dimer model on $\mathbb{Z}^2$, isoradial graphs

(?) General weighted planar bipartite graphs [Chelkak, Laslier, R.]

Definition: circle pattern

[Kenyon, Lam, Ramassamy, R.]



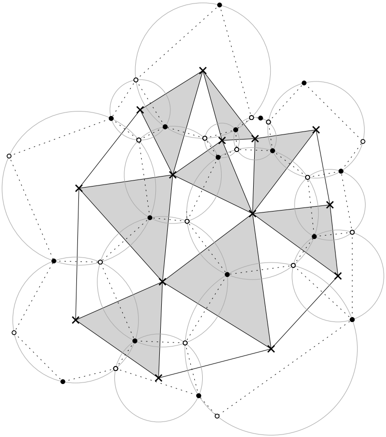
An embedding of a bipartite graph with cyclic faces.

Assume that each bounded face contains its circumcenter.

The circumcenters form an embedding of the dual graph.

Definition: circle pattern

[Kenyon, Lam, Ramassamy, R.]

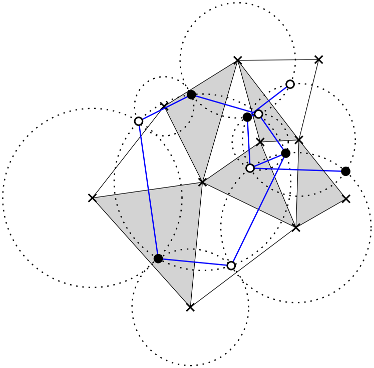


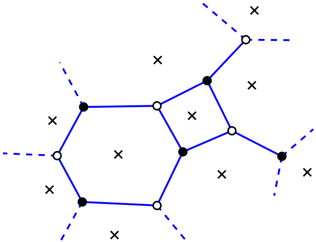
Circle pattern realisations with an embedded dual, where the dual graph is the graph of circle centres.

(!) Circle patterns themselves are not necessarily embedded.

Circle pattern

A circle pattern realisation with an embedded dual.

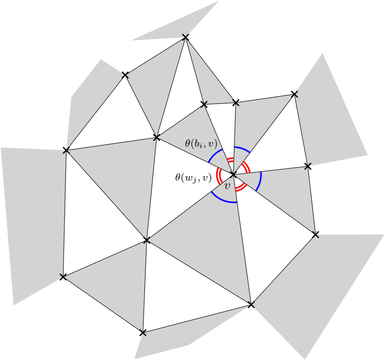




Definition: t-embedding

[Chelkak, Laslier, R.]

A t-embedding $\mathcal{T}$:

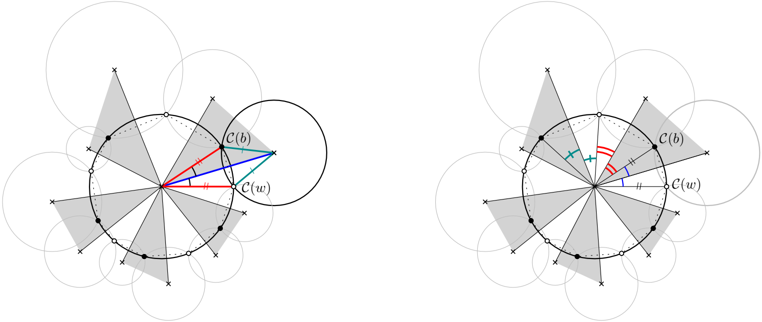


- **Proper:** All edges are straight segments and they don't overlap.
- **Bipartite dual:** The dual graph of $\mathcal{T}$ is bipartite.
- **Angle condition:** For every vertex v one has

$$\sum_{f \text{ white}} \theta(f, v) = \sum_{f \text{ black}} \theta(f, v) = \pi,$$

where $\theta(f, v)$ denotes the angle of a face f at the neighbouring vertex v.

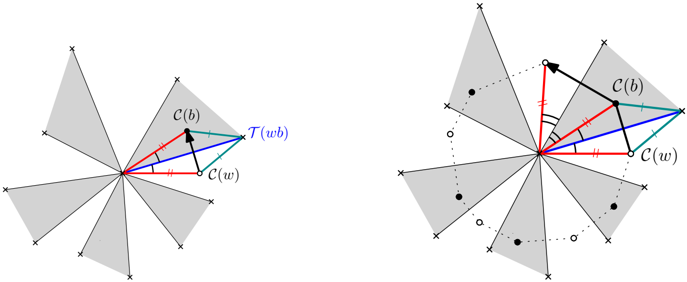
Circle pattern = t-embedding



Proposition (Kenyon, Lam, Ramassamy, R.)

Suppose $\mathcal{G}$ is a bipartite graph and $u : V(\mathcal{G}^*) \to \mathbb{C}$ is a convex embedding of the dual graph (with the outer vertex at ∞). Then there exists a circle pattern $\mathcal{C} : V(\mathcal{G}) \to \mathbb{C}$ with u as centers if and only if the alternating sum of angles around every dual vertex is 0.

Circle pattern = t-embedding



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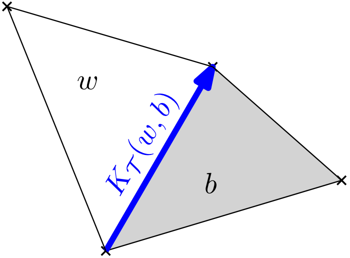
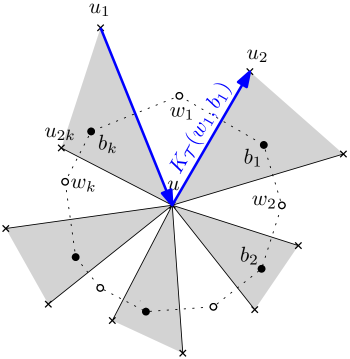
Kasteleyn weights

$$\mathcal{T} \to (\mathcal{G}, K_{\mathcal{T}}), \quad \text{where} \quad \sum_b K_{\mathcal{T}}(w,b) = \sum_w K_{\mathcal{T}}(w,b) = 0$$



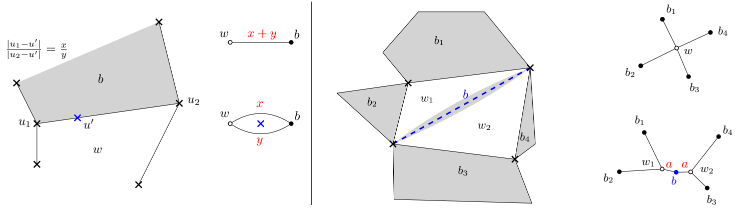


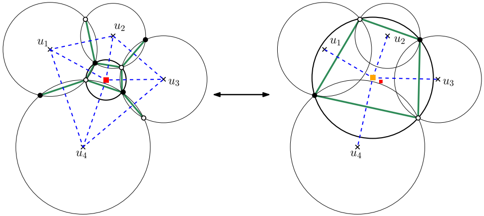
Then $K_{\mathcal{T}}$ is a Kasteleyn matrix.

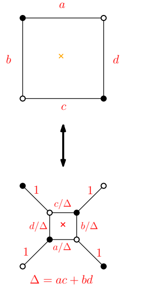
$\boxed{\text{Kasteleyn sign condition}} \longleftrightarrow \boxed{\text{angle condition}}$

$\prod \frac{K_{\mathcal{T}}(w_i,b_i)}{K_{\mathcal{T}}(w_{i+1},b_i)} \in (-1)^{k+1}\mathbb{R}_+$ $\qquad$ $\sum$ white $= \pi \mod 2\pi$

Circle patterns and elementary transformations



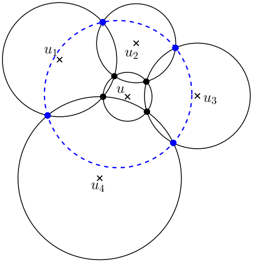




[Affolter; Kenyon, Lam, Ramassamy, R.]:
T-embeddings of $\mathcal{G}^*$ are preserved under elementary transformations of $\mathcal{G}$.

Circle patterns and elementary transformations

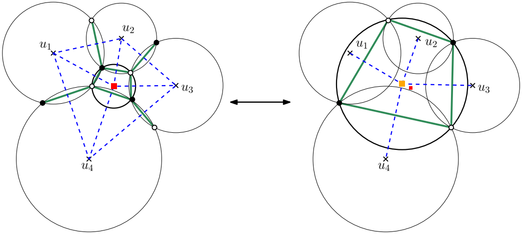
Miquel theorem:

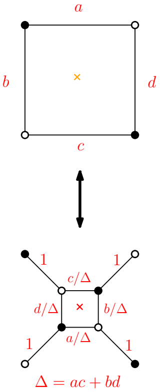


Central move

$$\frac{(u_2-u)(u_4-u)}{(u_1-u)(u_3-u)} = \frac{(u_2-\tilde{u})(u_4-\tilde{u})}{(u_1-\tilde{u})(u_3-\tilde{u})}$$

Circle patterns and elementary transformations



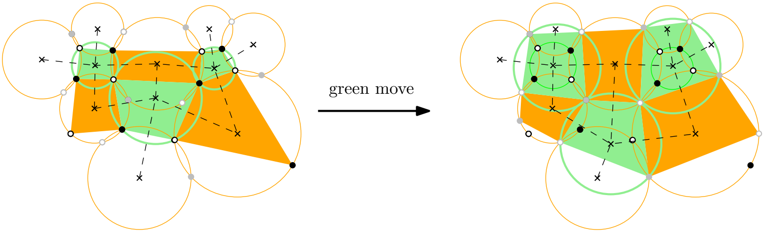


[Affolter; Kenyon, Lam, Ramassamy, R.]:

The Miquel move for circle centers corresponds to the urban renewal for dimer model.

Miquel dynamics on the square lattice

- Miquel dynamics defined as a discrete-time dynamics on the space of square-grid circle patterns: alternate Miquel moves on all the green faces then on all the orange faces.
- Its integrability follows from the identification with the Goncharov-Kenyon dimer dynamics.
- The evolution is governed by cluster algebras mutations.



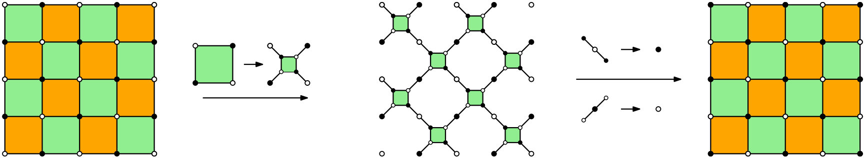
Miquel dynamics on the square lattice

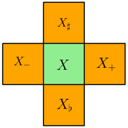
[Goncharov, Kenyon]:

Green move:

Step 1: Apply an urban renuval move to the green faces.

Step 2: Contract all the degree-2 vertices.





$$X \to X^{-1}$$

green move

$$X \to X \frac{(1+X_\sharp)(1+X_\flat)}{(1+X_+^{-1})(1+X_-^{-1})}$$

orange move

Existence of t-embeddings

$$(\mathcal{G}, K) \to (\mathcal{G}^*, K) \to (\mathcal{T}(\mathcal{G}^*), K_{\mathcal{T}}), \quad \text{where} \quad K \underset{\text{gauge}}{\sim} K_{\mathcal{T}}.$$

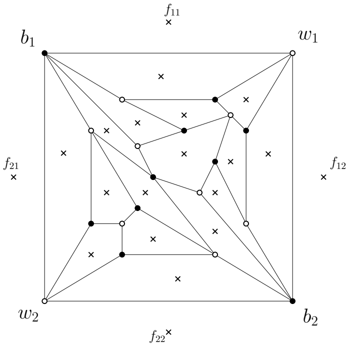
Theorem (Kenyon, Lam, Ramassamy, R.)

t-embeddings of the dual graph $\mathcal{G}^*$ exist at least in the following cases:

- *If $\mathcal{G}$ is a bipartite finite graph with outer face of degree 4, with an equivalence class of real Kasteleyn edge weights under gauge equivalence.*
- *If $\mathcal{G}$ is a biperiodic bipartite graph, with an equivalence class of biperiodic real Kasteleyn edge weights under gauge equivalence.*

$$K \underset{\text{gauge}}{\sim} K_{\mathcal{T}} \quad \longleftrightarrow \quad K_{\mathcal{T}}(wb) = G(w)K(wb)F(b)$$

Coulomb gauge for finite planar graphs



Def: Functions $G : W \to \mathbb{C}$ and $F : B \to \mathbb{C}$ are said to give Coulomb gauge for $\mathcal{G}$ if for all internal white vertices w

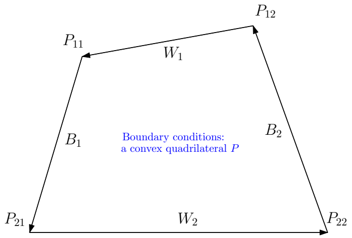
$$\sum_b G(w) K_{wb} F(b) = 0,$$

and for all internal black vertices b

$$\sum_w G(w) K_{wb} F(b) = 0.$$

$$\sum_w G(w) K_{wb_i} F(b_i) = B_i$$

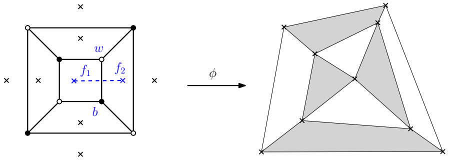
$$\sum_b G(w_i) K_{w_i b} F(b) = -W_i.$$

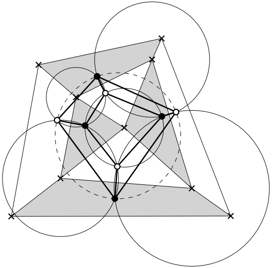


Coulomb gauge for finite planar graphs

Closed 1-form: $\omega(wb) = G(w)K_{wb}F(b)$.

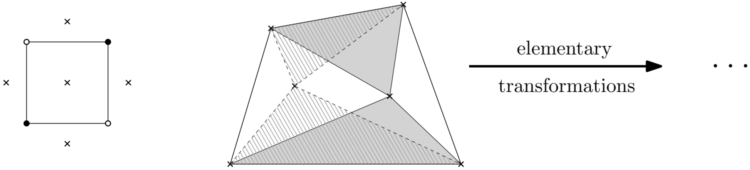
Define $\phi : \mathcal{G}^* \to \mathbb{C}$ by the formula $\phi(f_1) - \phi(f_2) = \omega(wb)$.





Theorem (Kenyon, Lam, Ramassamy, R.)

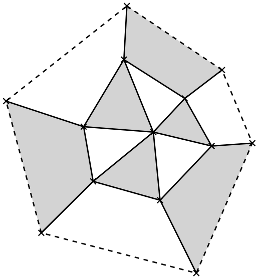
Suppose $\mathcal{G}$ has an outer face of degree 4. The mapping ϕ defines a convex t-embedding into P of $\mathcal{G}^*$ sending the outer vertices to the corresponding vertices of P.

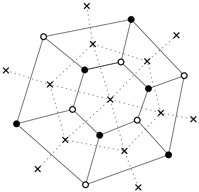


[A. Postnikov]:
Any nondegenerate planar bipartite graph with 4 marked boundary vertices w_1, b_1, w_2, b_2 can be built up from the 4-cycle graph with vertices w_1, b_1, w_2, b_2 using a sequence of elementary transformations; moreover the marked vertices remain in all intermediate graphs.

T-embeddings

Boundary of degree $2k$:





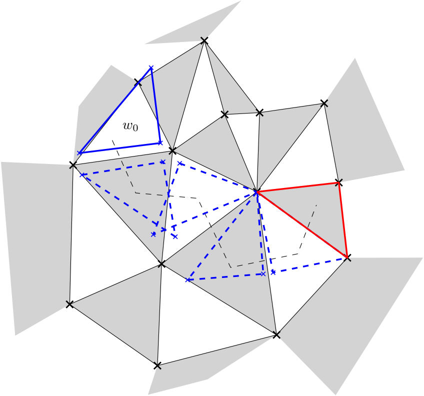
[Kenyon, Lam, Ramassamy, R.]:

- For each (generic) polygon P, there exists a t-embedding "realisation onto P".
- Usually not unique (finitely many)
- Maybe self-intersections.

Open question: Is it always a proper embedding?

Origami map

To get an origami map $\mathcal{O}(\mathcal{G}^*)$ from $\mathcal{T}(\mathcal{G}^*)$ one can choose a root face $\mathcal{T}(w_0)$ and fold the plane along every edge of the embedding.



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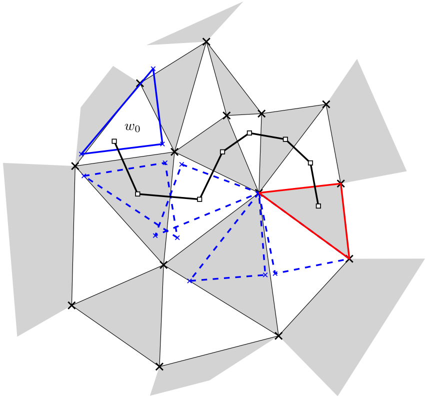
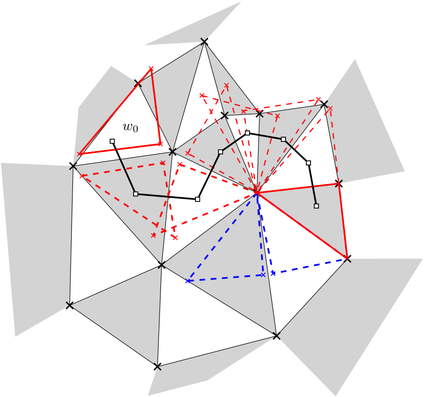


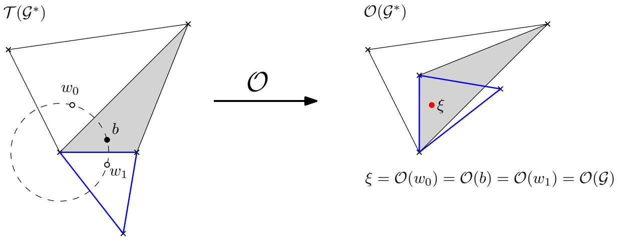
Origami map

To get an origami map $\mathcal{O}(\mathcal{G}^*)$ from $\mathcal{T}(\mathcal{G}^*)$ one can choose a root face $\mathcal{T}(w_0)$ and fold the plane along every edge of the embedding.



Uniqueness of biperiodic t-embeddings

To get an origami map $\mathcal{O}(\mathcal{G}^*)$ from $\mathcal{T}(\mathcal{G}^*)$ one can choose a root face $\mathcal{T}(w_0)$ and fold the plane along every edge of the embedding.



Theorem (Chelkak; Kenyon, Lam, Ramassamy, R.)

1. *The boundedness of the origami map $\mathcal{O}$ is equivalent to the boundedness of the radii in any circle pattern.*
2. *If $\mathcal{G}$ is biperiodic with biperiodic real Kasteleyn edge weights. There exists unique periodic t-embedding with a bounded $\mathcal{O}$.*

T-holomorphicity, assumptions

[Chelkak, Laslier, R.]

Assumption (**Lip(κ,δ)**)

Given two positive constant $\kappa < 1$ and $\delta > 0$ we say that a t-embedding $\mathcal{T}$ satisfies assumption $\mathrm{Lip}(\kappa,\delta)$ in a region $U \subset \mathbb{C}$ if $|\mathcal{O}(z') - \mathcal{O}(z)| \ \leq \ \kappa \cdot |z' - z|$ for all $z, z' \in U$ such that $|z - z'| \geq \delta$.

Remark:

- We think of δ as the 'mesh size';
- All faces have diameter less than δ;
- The actual size of faces could be in fact much smaller than δ.

T-holomorphicity, assumptions

[Chelkak, Laslier, R.]

Assumption (**Lip(κ,δ)**)

Given two positive constant $\kappa < 1$ and $\delta > 0$ we say that a t-embedding $\mathcal{T}$ satisfies assumption $\mathrm{L{\scriptsize IP}}(\kappa,\delta)$ in a region $U \subset \mathbb{C}$ if
$|\mathcal{O}(z') - \mathcal{O}(z)| \ \leq \ \kappa \cdot |z' - z|$ *for all $z, z' \in U$ such that $|z - z'| \geq \delta$.*

Assumption (**Exp-Fat(δ), triangulations**)

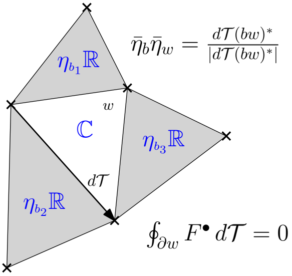
A sequence $\mathcal{T}^\delta$ of t-embeddings with triangular faces satisfes assumption $\mathrm{E{\scriptsize XP}}\text{-}\mathrm{F{\scriptsize AT}}(\delta)$ on a region $U^\delta \subset \mathbb{C}$ as $\delta \to 0$ if the following is fulfilled for each $\beta > 0$:
If one removes all '$\exp(-\beta\delta^{-1})$-fat' triangles from $\mathcal{T}^\delta$, then the size of remaining vertex-connected components tends to zero as $\delta \to 0$.

T-holomorphicity

[Chelkak, Laslier, R.]

- t-holomorphicity:
  Fix $\widetilde{w} \in W$. Given a function $F^\bullet_{\widetilde{w}}$ on B, s.t. $F^\bullet_{\widetilde{w}}(b) \in \eta_b\mathbb{R}$ and $K_{\mathcal{T}} F^\bullet_{\widetilde{w}} = 0$ at w, there exists $F^\circ_{\widetilde{w}}$ such that
  $F^\bullet_{\widetilde{w}}(b_i)$ are projections of $F^\circ_{\widetilde{w}}(w)$

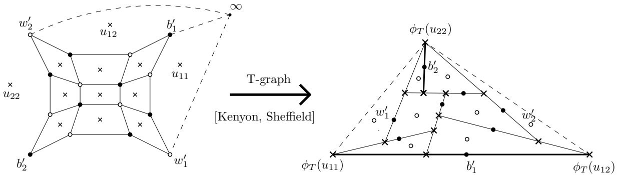
- bounded t-holomorphic functions are uniformly (in δ) Hölder and their contour integrals vanish as $\delta \to 0$.



$K^{-1}_{\mathcal{T}}(\,\cdot\,, w_0)$ is a t-holomorphic function for a fixed white vertex w_0

T-graph = t-embedding + Origami map

[Kenyon-Sheffield]: A pairwise disjoint collection $L_1, L_2, \dots, L_n$ of open line segments in $\mathbb{R}^2$ forms a T-graph in $\mathbb{R}^2$ if $\cup_{i=1}^n L_i$ is connected and contains all of its limit points except for some set of boundary points.



[Chelkak, Laslier, R.]:

- For any α with $|\alpha| = 1$, the set $\mathcal{T} + \alpha\mathcal{O}$ is a T-graph, possibly non proper and with degenerate faces.
- A t-white-holomorphic function F_w, can be integrated into a real harmonic function on a T-graph ($\mathrm{Re}(\mathrm{I}_{\mathbb{C}}[F_w])$ is harmonic on $\mathcal{T} + \mathcal{O}$).
- Lipschitz regularity of harmonic functions on $\mathcal{T} + \alpha\mathcal{O}$.

Height function $\to$ GFF

Theorem (Chelkak, Laslier, R.)

Assume that $\mathcal{T}^\delta$ satisfy assumptions $\mathrm{LIP}(\kappa,\delta)$ *and* $\mathrm{EXP\text{-}FAT}(\delta)$ *on compact subsets of* Ω *and*

(I) *The origami map is small:* $\mathcal{O}^\delta(z) \underset{\delta\to 0}{\longrightarrow} 0$

(II) $K^{-1}_{\mathcal{T}^\delta}(b^\delta, w^\delta)$ *is uniformly bounded as* $\delta \to 0$

(III) *the correlations* $\mathbb{E}[\hbar^\delta(v_1^\delta)\ldots\hbar^\delta(v_n^\delta)]$ *are uniformly small near the boundary of* Ω

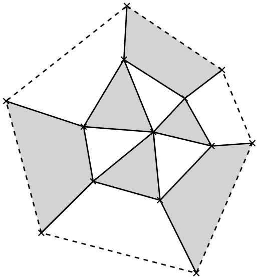
$\Rightarrow$ *convergence to* $\pi^{-1/2}\,\mathrm{GFF}_{\mathbb{D}}$.

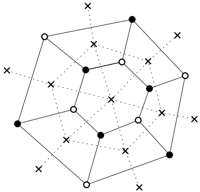
A similar (though more involved) analysis can be performed assuming that the origami maps $\mathcal{O}^\delta \underset{\delta\to 0}{\longrightarrow} \vartheta$, which is a graph of a Lorenz-minimal surface in $\mathbb{R}^{2+2}$.

[Chelkak, Laslier, R.]: *"Bipartite dimer model: perfect t-embeddings and Lorentz-minimal surfaces" (In preparation)*

T-embeddings

Boundary of degree $2k$:





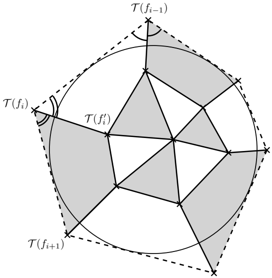
[Kenyon, Lam, Ramassamy, R.]:

- For each (generic) polygon P, there exists a t-embedding "realisation onto P".
- Usually not unique (finitely many)
- Maybe self-intersections.

Open question: Is it always a proper embedding?

Perfect t-embeddings

[Chelkak, Laslier, R.]



Definition. Perfect t-embeddings:

- P tangental to $\mathbb{D}$ [not necessary convex]
- $\mathcal{T}(f_i)\mathcal{T}(f_i')$ bisector of the $\mathcal{T}(f_{i-1})\mathcal{T}(f_i)\mathcal{T}(f_{i+1})$

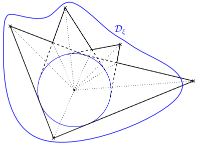
Remark:

- proper embeddings (no self-intersections) [at least if P is convex]
- Not unique:
  $(F, G) \rightsquigarrow$ perfect t-embedding, then
  for all $|\tau| < 1$
  $(F + \tau\bar{F}, G + \tau\bar{G}) \rightsquigarrow$ perfect t-embedding.

Open question: existence of perfect t-embeddings.

Conjecture: perfect t-embedding always exists.

Generalization



Theorem (Chelkak, Laslier, R.)

Let $\mathcal{G}^\delta$ be finite weighted bipartite pnanar graphs. Assume that

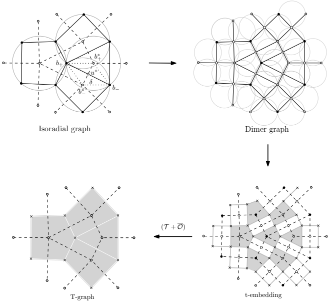
- $\mathcal{T}^\delta$ *are perfect t-embeddings of* $(\mathcal{G}^\delta)^*$ *satisfying assumption* $\textsc{Exp-Fat}(\delta)$
- $(\mathcal{T}^\delta, \mathcal{O}^\delta)$ *converge to a Lorentz-minimal surface* S.

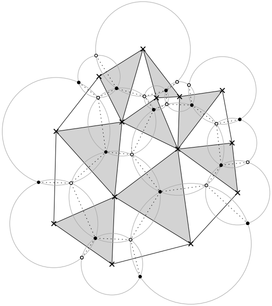
Then the height fluctuations converge to the standard Gaussian Free Field in the intrinsic metric of S.

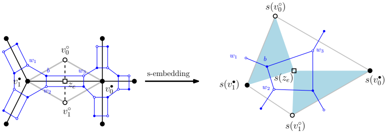
Chelkak, Laslier, R. *"Bipartite dimer model: perfect t-embeddings and Lorentz-minimal surfaces" (In preparation)*

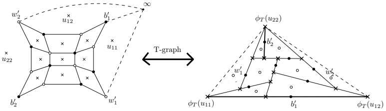
Chelkak, Ramassamy *"Fluctuations in the Aztec diamonds via a Lorentz-minimal surface" (arXiv:2002.07540)*

Thank you!


Isoradial graph
Dimer graph
$(\mathcal{T}+\overline{\mathcal{O}})$
T-graph
t-embedding




s-embedding
$s(v_0^\circ)$
$s(v_1^\bullet)$
$s(z_e)$
$s(v_0^\bullet)$
$s(v_1^\circ)$


∞
T-graph
$\phi_T(u_{22})$
$\phi_T(u_{11})$
$\phi_T(u_{12})$