Grassmannian categories of infinite rank

joint with Jenny August, Man-Wai Cheung, Eleonore Faber and Sira Gratz

Sibylle Schroll
University of Leicester

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Grassmannian categories of infinite rank

Idea: Categorify Grassmannian cluster algebras of infinite rank

Fomin-Zelevinsky 2002: A cluster algebra $\mathcal{A}$ is a subalgebra of $\mathbb{Z}[X_1^{\pm}, \ldots, X_n^{\pm}]$

- generators: cluster variables $\rightsquigarrow$ clusters
- $\mathcal{A}$ generated by mutation of clusters

Grassmannian cluster algebras of finite rank

$\mathrm{Gr}(k,n)$ Grassmannian of k-subspaces of $\mathbb{C}^n$

Theorem (Scott 2006)

$\mathbb{C}[Gr(k,n)]$ *has the structure of a cluster algebra.*

$$\mathbb{C}\,[\; p_I \mid I \subset \{1,\ldots,n\}, |I| = k \;] \,/\, \mathcal{I}_P$$

where

$$\mathcal{I}_P = \langle \sum_{r=0}^{k} (-1)^r p_{J' \cup \{j_r\}} p_{J \setminus \{j_r\}} \mid \\ J, J' \subset [n], |J| = k+1, |J'| = k-1, J = \{j_0, \ldots, j_k\} \rangle$$

Grassmannian cluster algebras of finite rank

Definition

Let I, J be two k-subsets of $\mathbb{Z}$.

- I and J are crossing if there are $i_1, i_2 \in I \setminus J$ and $j_1, j_2 \in J \setminus I$ such that

$$i_1 < j_1 < i_2 < j_2 \quad \text{or} \quad j_1 < i_1 < j_2 < j_2$$

- The Plücker coordinates p_I and p_J are compatible if I and J are non-crossing.

Grassmannian cluster algebras of finite rank

Theorem (Scott 2006)

Maximal sets of compatible Plücker coordinates are (examples of) clusters.

Example

<u>k = 2</u>

$$\text{Plücker coordinates} \overset{1-1}{\longleftrightarrow} \text{cluster variables}$$

$$\text{Plücker relations} \overset{1-1}{\longleftrightarrow} \text{exchange formulas}$$

Grassmannian cluster categories of finite rank

Jensen-King-Su 2016:

Categorification of Grassmannian cluster algebras of finite rank:

Set $\quad R_n = \mathbb{C}[x, y]/(x^k - y^{n-k})$

The group $\mu_n = \{\xi \in \mathbb{C} \mid \xi^n = 1\} < \mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{C}[x, y]$ by $x \mapsto \xi x$ and $y \mapsto \xi^{-1} y$

$MCM_{\mu_n}(R_n) := \mu_n$-equivariant maximal Cohen-Macaulay R_n-modules

Grassmannian cluster categories of finite rank

Theorem (Jensen-King-Su 2016)

$MCM_{\mu_n}(R_n)$ is a Frobenius category and

- $$\text{rank 1 modules} \stackrel{1-1}{\longleftrightarrow} \text{Plücker coordinates}$$
  $$M_I \longleftrightarrow p_I$$
- $Ext^1(M_I, M_J) = 0 \Longleftrightarrow p_I$ *and* p_J *are compatible*
- *Maximal sets of compatible Plücker coordinates correspond to cluster-tilting subcategories*
- *define a cluster character (using the categorification of the affine open cell via the pre-projective algebra by Geiss-Leclerc-Schröer)*

Grassmannian cluster algebras of infinite rank

Set $\mathcal{A}_k = \mathbb{C}[p_I \mid I \subset \mathbb{Z}, |I| = k]/\mathcal{I}_P$

Theorem (Grabowski-Gratz 2014)

$\mathcal{A}_k$ can be endowed with the structure of an infinite rank cluster algebra in uncountably many ways.

Theorem (Gratz 2015)

$\mathcal{A}_k$ is the colimt of cluster algebras of finite rank in the category of rooted cluster algebras.

Theorem (Groechening 2014)

Construction of $\mathcal{A}_k$ as coordinate ring of an infinite rank Grassmannian

$\rightsquigarrow$ $\underline{k = 2:}$ $\mathcal{A}_k$ is the homogeneous coordinate ring of an 'infinite' Grassmannian, the 2-dimensional subspaces of a profinite-dimensional vector space

Grassmannian categories of infinite rank

Idea: $n \to \infty$ in $\mathrm{Gr}(k,n)$ and $x^k - y^{n-k}$

$$\rightsquigarrow \mathrm{Gr}(k,\infty) \text{ and } R := \mathbb{C}[x,y]/x^k$$

$\mathbb{G}_m = \mathbb{C}^*$ acts on $\mathbb{C}[x,y]$ by $x \mapsto \xi x$ and $y \mapsto \xi^{-1} y$ for $\xi \in \mathbb{G}_m$

$\mathrm{MCM}_{\mathbb{G}_m} R$:= $\mathbb{G}_m$-equivariant maximal Cohen-Macaulay modules

Since $\mathrm{Hom}(\mathbb{G}_m, \mathbb{C}) \simeq \mathbb{Z}$, we have

$$\mathrm{mod}_{\mathbb{G}_m} R \simeq \mathrm{gr}\, R$$

$\rightsquigarrow$

$$\mathrm{MCM}_{\mathbb{G}_m} R \simeq \mathrm{MCM}_{\mathbb{Z}} R$$

The category of $\mathbb{Z}$-graded maximal Cohen-Macaulay R-modules is a Grassmannian category of infinite rank.

Grassmannian categories of infinite rank

$\mathrm{MCM}_{\mathbb{Z}}R$ is a Frobenius category

Theorem (Buchweitz 1986)

$$\underline{MCM_{\mathbb{Z}}}R \simeq D_{sg}(gr\,R)$$

<u>k = 2</u>: Holm-Jørgensen 2012: The derived category with finite cohomology $D^f_{dg}(\mathbb{C}[y])$ of the differential graded algebra $\mathbb{C}[y]$ with $deg(y) = -1$ has cluster combinatorics of type A.

Remark:
Set $\mathcal{C} = \langle$ generically free rank 1 $\mathrm{MCM}_{\mathbb{Z}}\mathbb{C}[x,y]/x^2$ modules $\rangle$.
Then $\underline{\mathcal{C}} \simeq D^f_{dg}(\mathbb{C}[y])$.

Yildirm-Paquette 2020: Completion of discrete cluster categories of infinite type by Igusa-Todorov (2015).

$\rightsquigarrow$ for $k = 2$ and with 1 accumulation point :

$$\text{Yildirim-Paquette completion} \simeq \mathrm{MCM}_{\mathbb{Z}}\mathbb{C}[x,y]/x^2$$

Generically free modules

Set $\mathcal{F} = \mathbb{C}[x, y]/x^k$ total ring of fractions

Definition

A module M in $\mathrm{MCM}_{\mathbb{Z}} R$ is generically free of rank n if $M \otimes_R \mathcal{F}$ is a graded free $\mathcal{F}$-modules of rank n.

Proposition

1. *If $M \in MCM_{\mathbb{Z}} R$ is generically free then $M = \Omega(N)$ for some finite dimensional $N \in gr\, R$.*
2. *$M \in MCM_{\mathbb{Z}} R$ is generically free of rank 1 $\iff$ M is a graded ideal of R and $y^n \in M$ for some $n > 0$.*
3. *Every homogeneous ideal I of R can be generated by monomials.*

Generically free rank 1 modules

Theorem (August-Cheung-Faber-Gratz-S. 2020)

A module M in $MCM_{\mathbb{Z}}R$ is generically free of rank 1
$\iff M = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \ldots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$
with $0 \leq i_1 \leq i_2 \leq \cdots \leq i_{k-1}$ *and* $i_k \in \mathbb{Z}$.

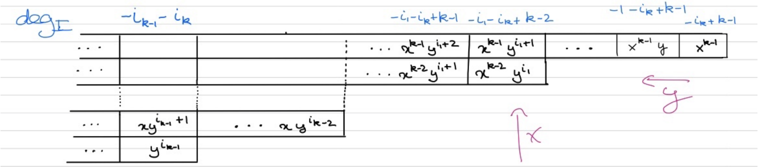


Figure: Schematical view of a rank 1 module.

Definition

Define the strictly non-decreasing degree sequence to be
$\ell_I := (-i_{k-1} - i_k, -i_{k-2} - i_k + 1, \ldots, -i_k + k - 1)$

Generically free rank 1 modules and Plücker coordinates

Corollary

$$\left\{ \begin{array}{c} \textit{generically free rank 1} \\ \textit{modules in } MCM_{\mathbb{Z}}R \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Plücker coordinates} \\ \textit{in } \mathcal{A}_k \end{array} \right\}$$

$$I \longmapsto p_{\ell_I}$$

$$I(\ell) \longleftarrow \ell = (\ell_1, \ldots, \ell_k)$$

where $I(\ell) = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \ldots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$ *with* $i_k = k - 1 - \ell_k$ *and* $i_{k-r} = \ell_k - \ell_r - k + r$ *for* $1 \leq r \leq k - 1$.

Remark: This bijection is structure preserving.

Rigidity and compatibility

Theorem (August-Cheung-Faber-Gratz-S. 2020)

Let $I, J \in MCM_{\mathbb{Z}}R$ generically free of rank 1. Then

$$Ext^1(I,J) = 0 \iff p_{\ell_I} \text{ and } p_{\ell_J} \text{ are compatible}$$

$$\iff Ext^1(J,I) = 0$$

Corollary

Generically free rank 1 modules in $MCM_{\mathbb{Z}}R$ are rigid.

Idea of Proof:

I generically free $\mathrm{MCM}_{\mathbb{Z}} R$ module.
The matrix factorisation of I

$$R^k \xrightarrow{M} R^k \xrightarrow{N} R^k \longrightarrow I \longrightarrow 0$$

gives a graded projective presentation of I.
Apply graded $\mathrm{Hom}(-, J)$ noting that $\mathrm{Hom}(R(m), J) = J(-m)$

$$\rightsquigarrow \qquad \mathbf{J} \xrightarrow{N^\top} \mathbf{J}(1) \xrightarrow{M^\top} \mathbf{J}(k)$$

where $\mathbf{J}$ is a direct sum of appropriately shifted copies of J.

$$\rightsquigarrow \qquad \mathrm{Ext}^1(I, J) = \left(\mathrm{Ker} M^\top / \mathrm{Im} N^\top\right)_0 = \mathrm{Ker}(M^\top)_0 / \mathrm{Im}(N^\top)_0$$

Dimension formula

$$\dim \mathrm{Ext}^1(I,J) = \dim \mathrm{Ker}(M^\top)_0 - \dim \mathrm{Im}(N^\top)_0$$
$$\dim \mathrm{Ker}(M^\top)_0 = \dim \mathbf{J}(1)_0 - \dim \mathrm{Im}(M^\top)_0$$
$$\dim \mathrm{Im}(N^\top)_0 = \dim \mathbf{J}_0 - \dim \mathrm{Ker}(N^\top)_0$$

We then show

$$\dim \mathbf{J}_0 - \dim \mathbf{J}(1)_0 = |\ell_I \cap \ell_J|$$
$$\dim \mathrm{Im}(M^\top)_0 = k - \beta(\ell_I, \ell_J)$$
$$\dim \mathrm{Ker}(N^\top)_0 = \alpha(\ell_I, \ell_J)$$

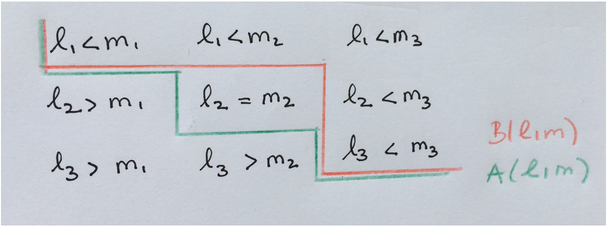
Theorem (August-Cheung-Faber-Gratz-S. 2020)

$$\begin{aligned} \mathit{dim}\ \mathit{Ext}^1(I,J) &= \alpha(\ell_I,\ell_J) + \beta(\ell_I,\ell_J) - k - |\ell_I \cap \ell_J| \\ &= \mathit{dim}\ \mathit{Ext}^1(J,I) \end{aligned}$$

New combinatorial tool: staircase paths

Example of calculation of dim $\mathrm{Ext}^1(I,J)$:

$$k=3: \begin{array}{lllllll} \ell_I & = & (-5,1,3) & = & (\ell_1,\ell_2,\ell_3) & = & \ell \\ \ell_J & = & (0,1,4) & = & (m_1,m_2,m_3) & = & m \end{array}$$



$\alpha(\ell,m) = \#$ diagonals strictly above $A(\ell,m) = 3$
$\beta(\ell,m) = \#$ diagonals strictly below $B(\ell,m) = 2$

$|\ell \cap m| = 1 \implies \dim \mathrm{Ext}^1(I,J) = 3+2-3-1 = 1$

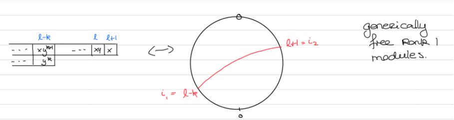
with $I = (x^2, xy, y^6)(-1)$ and $J = (x^2, xy^2, y^2)(-2)$ in $\mathrm{MCM}_{\mathbb{Z}} R$.

k=2

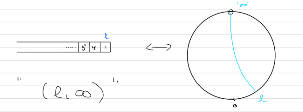
Proposition

The indecomposable $MCM_{\mathbb{Z}}(\mathbb{C}[x,y]/x^2)$ modules correspond to

- $(x, y^k)(-\ell)$



- $\mathbb{C}[y](-\ell)$



Two arcs γ, δ corresponding to $I(\gamma), I(\delta) \in MCM_{\mathbb{Z}}(\mathbb{C}[x,y]/x^2)$
dim $Ext^1(I(\alpha), I(\beta)) = 1 \iff \gamma$ and δ cross (possibly at ∞).
dim $Ext^1(I(\alpha), I(\beta)) = 0 \iff \gamma$ and δ do not crossing.

Cluster tilting subcategories

We can completely describe the Hom-spaces between indecomposables

Theorem (August-Cheung-Faber-Gratz-S. 2020)

$MCM_{\mathbb{Z}}(\mathbb{C}[x,y]/x^2)$ has cluster tilting subcategories and they are of the form