Perturbing Isoradial Triangulations

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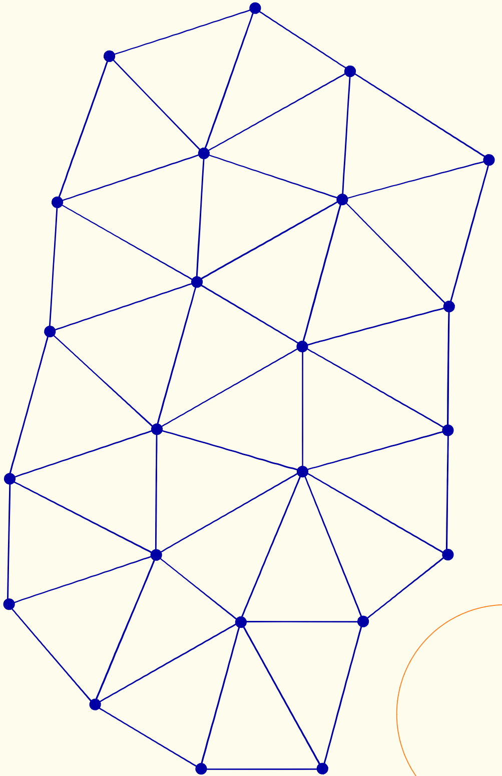
$\boxed{\text{heuristic}}$

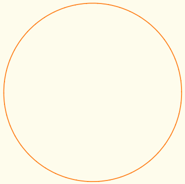
discrete world		continuous world
moduli-space of Delaunay triangulations $\subset \ Gr_{\mathbb{C}}(2,\infty) \,/\, \infty\text{-torus}$	$\overset{\text{correspondence}}{\Longleftrightarrow}$	space of metrics $g=(g_{\mu\nu})$ on the plane
$\cup$		$\cup$
isoradial triangulations	$\Longleftrightarrow$	flat metrics

(fragment of an) infinite <u>Delaunay</u> <u>triangulation</u> T

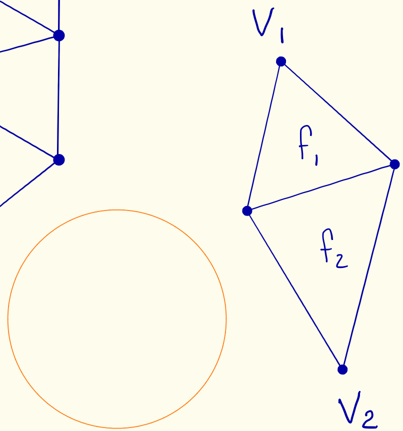
i.e infinite planar graph <u>embedded</u> in $\mathbb{C}$ with vertices, edges, faces

$V(T)$ $\quad E(T)$ $\quad F(T)$



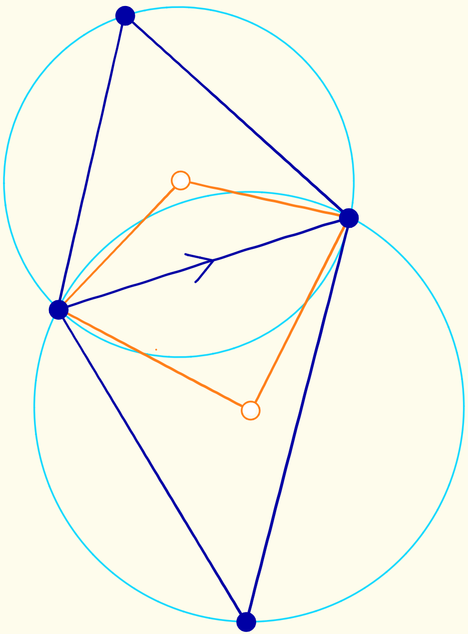


- each face $f \in F(T)$ triangle
- faces cover $\mathbb{C}$, locally finite
- for all abutting triangles f_1, f_2



$V_1 \notin$ interior of the disk of circumcircle C_2

$V_2 \notin$ interior of the disk of circumcircle C_1



Notation / set-up :

for $u \in V(T)$, $z(u) \in \mathbb{C}$ coordinate

for an oriented edge $\overrightarrow{uv}$ have

- north and south triangles X_n, X_s
- circumcenters x_n, x_s
- circumradii R_n, R_s
- angles θ_n, θ_s

def the conformal angle

$$\theta(uv) = \frac{1}{2}\left(\theta_n + \theta_s\right)$$

(symmetric, $SL_2(\mathbb{C})$ - invariant) three operators Θ acting on $\mathbb{C}[T] = \{\phi : V(T) \to \mathbb{C}\}$ each having the form

$$\boxed{\Theta\phi(u) = \sum_{\substack{u \in V(T) \\ \overrightarrow{uv}}} c_\Theta(\overrightarrow{uv})\left[\phi(u) - \phi(v)\right]}$$

(1) Beltrami-Laplace operator Δ

$$c_\Delta(\overrightarrow{uv}) = \frac{1}{2}\left[\tan\theta_n(\overrightarrow{uv}) + \tan\theta_s(\overrightarrow{uv})\right]$$

(2) discrete Kähler operator $\mathscr{D}$

$$c_{\mathscr{D}}(\overrightarrow{uv}) = \frac{1}{2}\left[\frac{\tan\theta_n(\overrightarrow{uv}) + i}{R_n^2} + \frac{\tan\theta_s(\overrightarrow{uv}) - i}{R_s^2}\right]$$

(3) conformal Laplacian Δ_{conf}

$$c_{\Delta_{conf}}(\overrightarrow{uv}) = \tan\theta(\overrightarrow{uv})$$

<u>def</u> a Delaunay triangulation T is <u>isoradial</u> if all circumradii are equal; $R_f = R_{cr} > 0 \quad \forall f \in F(T)$

$$\boxed{\begin{array}{l} \text{in this case} \quad \underbrace{\Delta = \mathcal{D} = \Delta_{conf}} \\ (\text{and } R_{cr} = 1) \quad \text{"critical laplacian" } \Delta_{cr} \end{array}}$$

Green's function Δ_{cr}^{-1} (studied by R. Kenyon '02) characterised by properties

(1) $\Delta_{cr} \Delta_{cr}^{-1} = \mathbb{1}$ (means that $v \longmapsto [\Delta_{cr}^{-1}]_{uv}$ is harmonic)

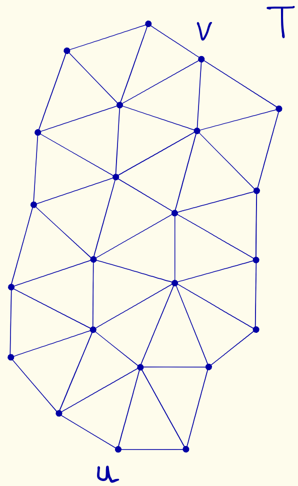
(2) $[\Delta_{cr}^{-1}]_{uv} = O(\log |z_u - z_v|)$
for $u, v \in V(T)$, $|z_u - z_v| \gg 0$

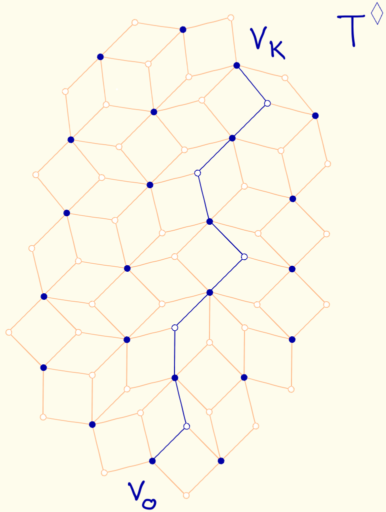
(3) $[\Delta_{cr}^{-1}]_{u,u} = 0$

by [Kenyon 2002]

(1) $$[\Delta_{cr}^{-1}]_{uv} = \frac{1}{2\pi} \operatorname{re} \int_0^1 \frac{dt}{t} \left(E_{\underline{\theta}}(-t) - 1 \right)$$

where $$E_{\underline{\theta}}(z) = \prod_{j=1}^{K} \frac{z + e^{i\theta_j}}{z - e^{i\theta_j}}$$





$v_0 = u$ and $v_K = v$

$e^{i\theta_j} = z(v_j) - z(v_{j-1})$

rhombic graph

$v_0, \dots, v_K$ path

(2) Asymptotics $[\Delta_{cr}^{-1}]_{uv} =$

$$-\frac{1}{2\pi}\left(\log 2D + \gamma_{euler} + \frac{re[P_3(u,v)]}{6D^3} + O(D^{-4})\right)$$

here $D = |z(u) - z(v)|$ and $|P_3(u,v)| \leq 3D$

for $\epsilon_1, \dots, \epsilon_m > 0$ small
T remains Delaunay
(for all values $0 < \ell \leq \infty$)

↑ in general <u>not</u> isoradial

<u>Perturbations:</u>

Fix an isoradial triangulation T_{cr}

<u>deform</u> <u>embedding</u>; for $v \in V(T_{cr})$

But keep incidence relations !

(i.e. same vertex, edge, face sets)

$$z(v) \longmapsto z(v) + \epsilon_1 \ell F_1\left(z(v)/\ell\right) + \dots + \epsilon_m \ell F_m\left(z(v)/\ell\right)$$

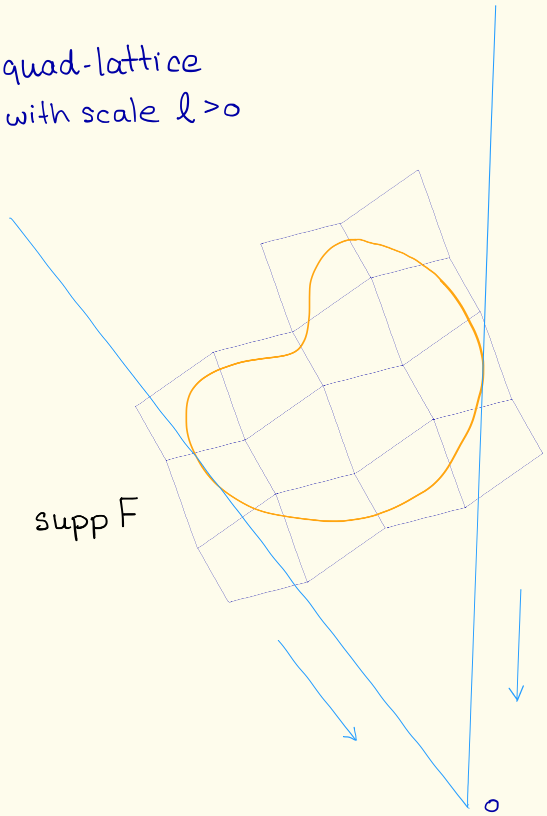
$\epsilon_1, \dots, \epsilon_m > 0$ deformation parameters

$\ell > 0$ scaling parameter

$F_1, \dots, F_m : \mathbb{C} \to \mathbb{C}$ smooth, compact supports

$\operatorname{supp} F_i \cap \operatorname{supp} F_j = \emptyset$ for all i,j

$m = 1$, i.e. one zone



mesh contracting towards origin as scaling parameter $\ell \to \infty$

set $\delta\Theta = \Theta - \Delta_{cr}$ and then formally expand

$$\log\det\Theta = \log\det\Delta_{cr} + \mathrm{tr}\left[\delta\Theta\,\Delta_{cr}^{-1}\right] - \frac{1}{2}\mathrm{tr}\left[\delta\Theta\,\Delta_{cr}^{-1}\right]^2 + \cdots$$

<u>Thm</u> [F. David + J. Scott] let $m=2$ (i.e. bilocal perturbation) and $\Theta = \Delta$ or $\mathcal{D}$ then the $\epsilon_1\epsilon_2$-crossterm of $\mathrm{tr}\left[\delta\Theta\,\Delta_{cr}^{-1}\right]^2$ is

$$-\frac{2}{2\pi^2}\sum_{\substack{\text{triangles}\\(X_1, X_2)}} A(X_1)A(X_2)\,\mathrm{re}\left[\frac{\bar\partial F_1(x_1/\ell)\,\bar\partial F_2(x_2/\ell)}{(x_1 - x_2)^4}\right] + O\left(D_\ell^{-5}\right)$$

where x_i is the center of X_i

$D_\ell = \mathrm{dist}\left(\Omega_1(\ell), \Omega_2(\ell)\right)$

and $\Omega_i(\ell) = \left\{z \in \mathbb{C} \,\middle|\, z/\ell \in \mathrm{supp}\, F_i\right\}$

connections with CFT :

$\boxed{\text{heuristic}}$

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$\delta g = J^T g + g J$

$J = \text{jac.}(F)$

$F: M \to M$

general riemannian manifold (M,g) (g flat)
equipped with Beltrami-Laplace Δ_g (self-adjoint)
with respect to pairing $\langle \phi, \psi \rangle = \int_M dx\, |g|^{1/2} \phi(x)\psi(x)$
by analogy with f.d. Gaussian integrals "define"

$$\det{}^{-1/2} \Delta_g = \int D[\phi] \exp -\frac{1}{2}\langle \phi, \Delta_g \phi \rangle =: Z_g$$

i.e. $-\frac{1}{2}\log\det \Delta_g = \log Z_g$

perturb $g \mapsto g + \epsilon \delta g$
do functional Taylor expansion
use moment formulae of the
formal gaussian integral to get

$$\log Z_{g+\epsilon\delta g} = \log Z_g + \epsilon \log Z_g^{(1)} + \epsilon^2 \log Z_g^{(2)} + \cdots$$

by CFT, $M = \mathbb{C}$

$$Z_g^{(2)} = \frac{-2}{4\pi^2} \iint_{\mathbb{C}\times\mathbb{C}} du\,dv\ \mathrm{re}\left[\frac{\bar\partial F(u)\,\bar\partial F(v)}{(u-v)^4}\right]$$

opening remarks:

this is joint work with François David at IPhT, Saclay
project is born out of the heuristic which sees
infinite triangulation of the plane (specifically Delaunay
triangulations) as discretisations of continuous metrics
on the plane: Just as a metric g determines
a Laplace-Beltrami operator Δ_g, likewise a Delaunay
triangulation carries a Laplace-Beltrami operator. And
just as one can formalise taking the determinant
$\det \Delta_g$ of a continuous LB-operator, a similiar
analysis can be used for triangulations.

side-step by looking at the difference.

Avoid explaining the mathematical scaffolding needed
to make sense of determinants of ∞-dim'l operators