Alternating sign matrices and the many faces of dynamical algebraic combinatorics

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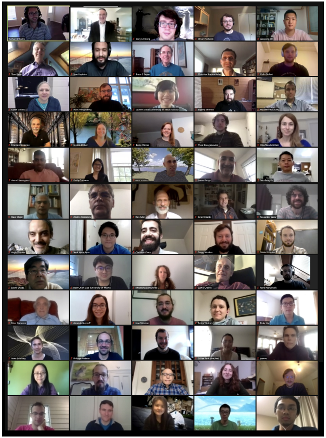
June 26, 2024

Dynamical algebraic combinatorics workshops

AIM (2015), BIRS online (2020), BIRS hybrid (2021)
Organizers: James Propp, Tom Roby, Jessica Striker, Nathan Williams



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Alternating-Sign Matrices and Domino Tilings (Part I)*

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Abstract. We introduce a family of planar regions, called Aztec diamonds, and study tilings of these regions by dominoes. Our main result is that the Aztec diamond of order n has exactly $2^{n(n+1)/2}$ domino tilings. In this, the first half of a two-part paper, we give two proofs of this formula. The first proof exploits a connection between domino tilings and the alternating-sign matrices of Mills, Robbins, and Rumsey. In particular, a domino tiling of an Aztec diamond corresponds to a compatible pair of alternating-sign matrices. The second proof of our formula uses monotone triangles, which constitute another form taken by alternating-sign matrices; by assigning each monotone triangle a suitable weight, we can count domino tilings of an Aztec diamond.

Keywords: tiling, domino, alternating-sign matrix, monotone triangle, representation, square ice

1. Introduction

The *Aztec diamond of order* n is the union of those lattice squares $[a, a+1] \times [b, b+1] \subset \mathbf{R}^2$ $(a, b \in \mathbf{Z})$ that lie completely inside the tilted square $\{(x, y) : |x| + |y| \le n + 1\}$. (Figure 1 shows the Aztec diamond of order 3.) A *domino* is a closed 1×2 or 2×1 rectangle in $\mathbf{R}^2$ with corners in $\mathbf{Z}^2$, and a *tiling* of a region R by dominoes is a set of dominoes whose interiors are disjoint and whose union is R. In this paper we will show that the number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$. We will furthermore obtain more refined enumerative information regarding two natural statistics of a tiling: the number of vertical tiles and the "rank" of the tiling (to be defined shortly).

Fix a tiling T of the Aztec diamond of order n. Every horizontal line $y = k$ divides the Aztec diamond into two regions of even area; it follows that the

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The Many Faces of Alternating-Sign Matrices

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I give a survey of different combinatorial forms of alternating-sign matrices, starting with the original form introduced by Mills, Robbins and Rumsey as well as corner-sum matrices, height-function matrices, three-colorings, monotone triangles, tetrahedral order ideals, square ice, gasket-and-basket tilings and full packings of loops.

Keywords: Alternating-Sign Matrices, Tilings

1 Introduction

An *alternating-sign matrix of order n* is an n-by-n array of 0's, $+1$'s and -1's with the property that in each row and each column, the non-zero entries alternate in sign, beginning and ending with a $+1$. For example, Figure 1 shows an alternating-sign matrix (ASM for short) of order 4.

$$\begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & -1 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & 0 & 0 \end{pmatrix}$$

Figure 1: An alternating-sign matrix of order 4.

Figure 2 exhibits all seven of the ASMs of order 3.

$$\begin{pmatrix} 0 & 0 & +1 \\ 0 & +1 & 0 \\ +1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & +1 \\ +1 & 0 & 0 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ +1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & +1 & 0 \\ +1 & -1 & +1 \\ 0 & +1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

Figure 2: The seven alternating-sign matrices of order 3.

†This work was supported by grants from the National Science Foundation and the National Security Agency.

Alternating sign matrices

Definition

Alternating sign matrices (ASM) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$
- each row and each column sums to 1
- nonzero entries alternate in sign along a row/column

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

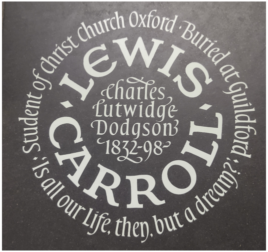
Alternating sign matrix enumeration

Theorem (Zeilberger 1996; Kuperberg 1996)

$n \times n$ alternating sign matrices are counted by: $$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

1, 2, 7, 42, 429, 7436, 218348, 10850216, ...

This was conjectured by Mills, Robbins, and Rumsey (1983) and proved in different ways by Zeilberger (1996), Kuperberg (1996), Fischer (2006), and Fischer-Konvalinka (2020/2022).

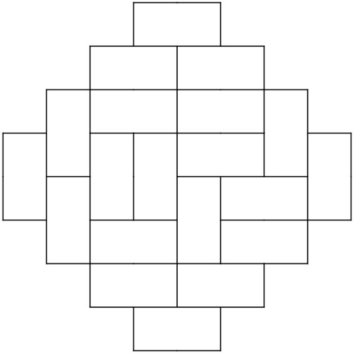


Alternating sign matrices and domino tilings

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the order n Aztec diamond is:

$$AD(n) = \sum_{A \in ASM_n} 2^{(\# -1s \text{ in } A)} = 2^{n(n-1)/2}.$$



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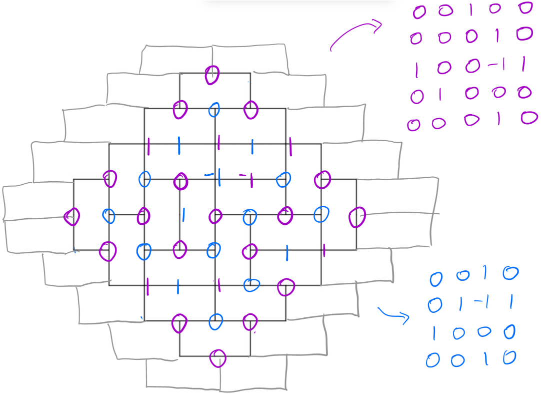


The many faces of alternating sign matrices

ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

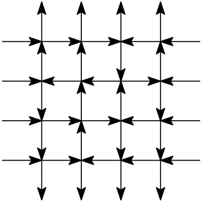
Monotone triangle

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

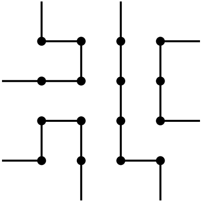
Height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

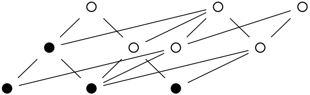
Six-vertex model



Fully-packed loop



Order ideal



Partial alternating sign matrix bijections and dynamics [Heuer 2024]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix}$$

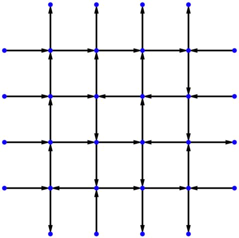
1

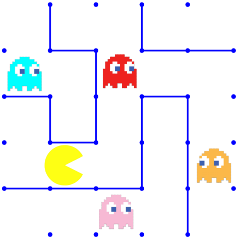
1 3

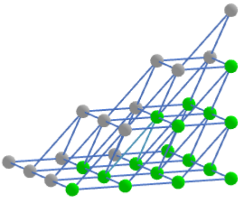
0 2 3

0 1 2 4

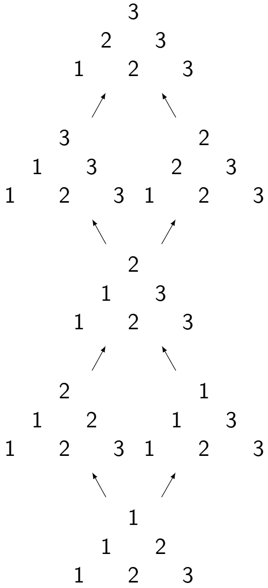
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 2 & 3 & 2 \\ 3 & 4 & 3 & 2 & 3 \\ 4 & 3 & 4 & 3 & 2 \end{pmatrix}$$







A partial order on alternating sign matrices

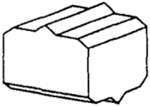


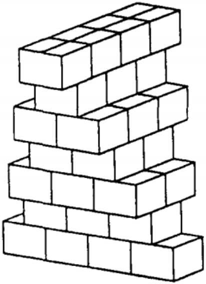
Tetrahedral order ideals

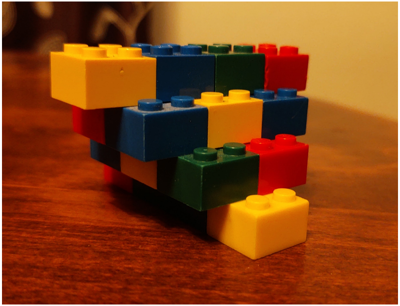
A partial order on alternating sign matrices

Theorem (Elkies, Kuperberg, Larsen, and Propp 1992)

The partial order on alternating sign matrices is a distributive lattice (that is, a lattice of order ideals) with a nice structure.

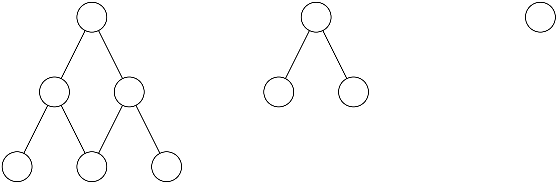




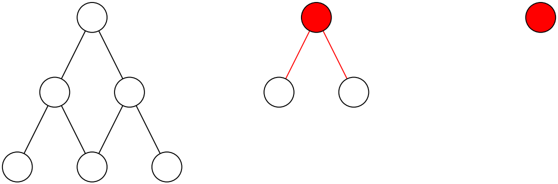


https://arxiv.org/pdf/math/9201305.pdf

Alternating sign matrix tetrahedral poset



Alternating sign matrix tetrahedral poset



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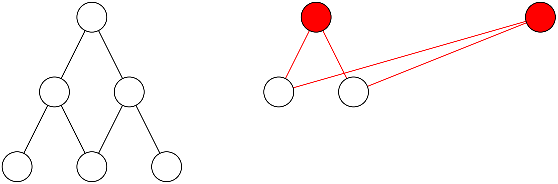

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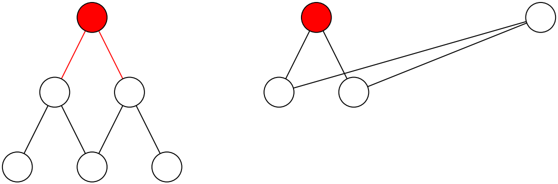

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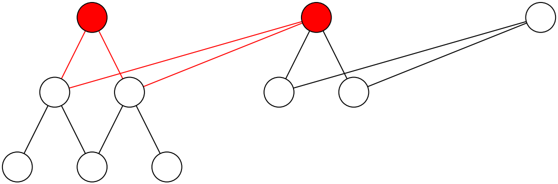

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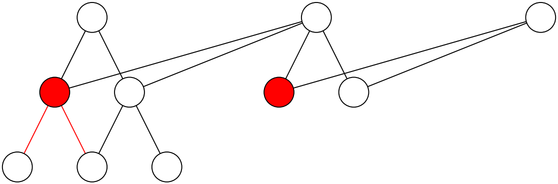

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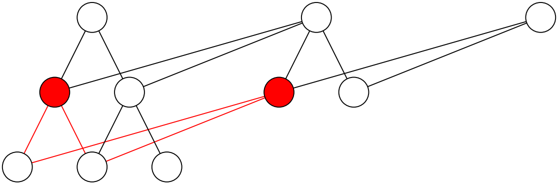

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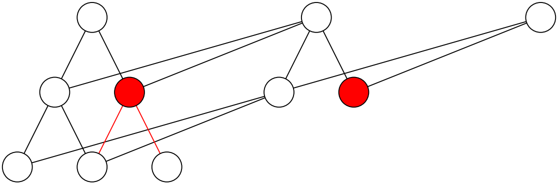

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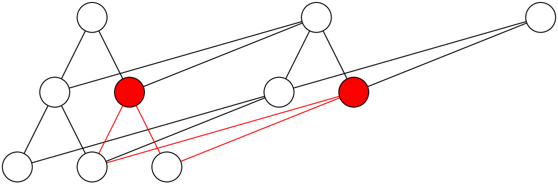

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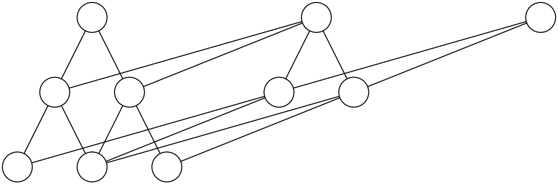

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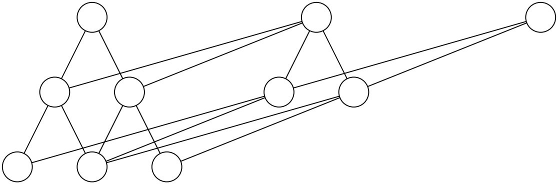
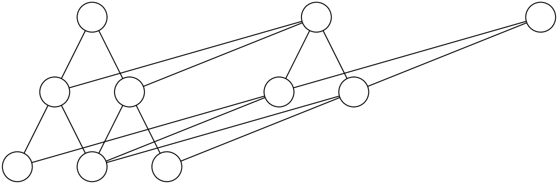


$n \times n$ alternating sign matrices are in bijection with order ideals in this $(n-1)$-layer poset.

Alternating sign matrix tetrahedral poset



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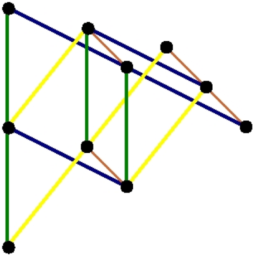

A unifying poset perspective on alternating sign matrices, plane partitions, Catalan objects, tournaments, and tableaux

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ABSTRACT

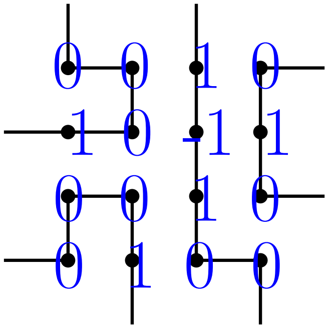
Alternating sign matrices (ASMs) are square matrices with entries 0, 1, or −1 whose rows and columns sum to 1 and whose nonzero entries alternate in sign. We present a unifying perspective on ASMs and other combinatorial objects by studying a certain tetrahedral poset and its subposets. We prove the order ideals of these subposets are in bijection with a variety of interesting combinatorial objects, including ASMs, totally symmetric self-complementary plane partitions (TSSCPPs), staircase shaped semistandard Young tableaux, Catalan objects, tournaments, and totally

Gyration and Resonance

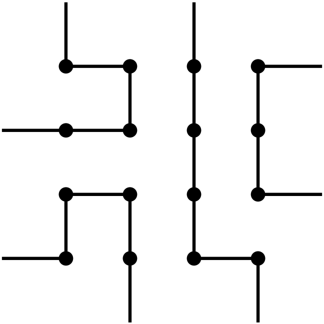
Alternating sign matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Alternating sign matrix ↔ fully-packed loop

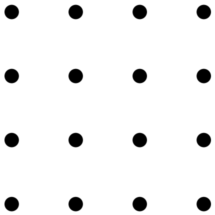


Fully-packed loop



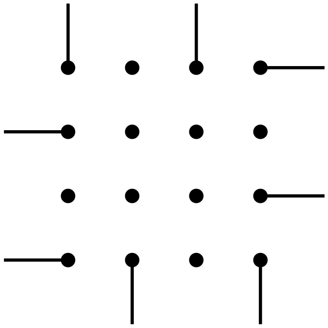
Fully-packed loops

Start with an $n \times n$ grid.



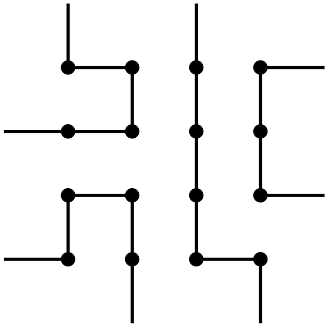
Fully-packed loops

Add boundary conditions.



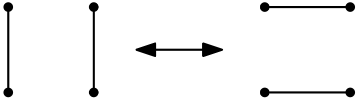
Fully-packed loops

Interior vertices adjacent to 2 edges.



Dynamics: Gyration on fully-packed loops

The local move.



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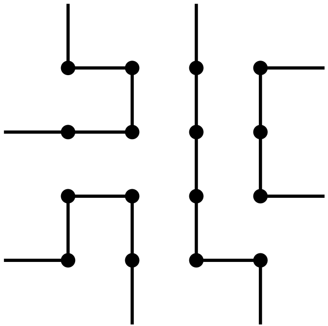
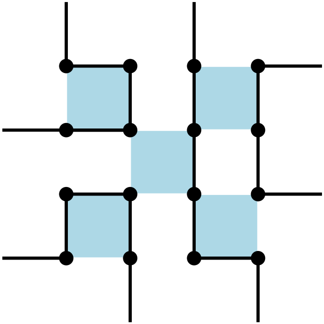


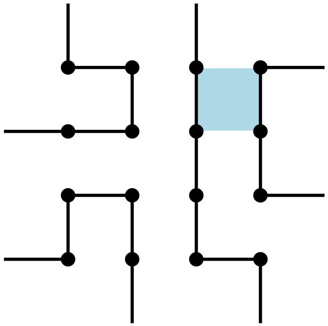
Dynamics: Gyration on fully-packed loops

Start with the even squares.



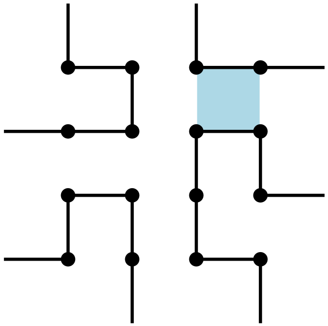
Dynamics: Gyration on fully-packed loops

Apply the local move.



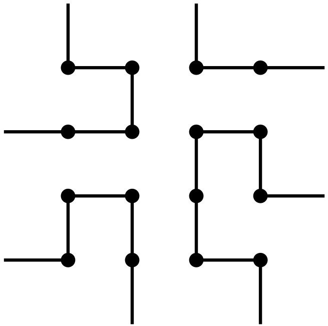
Dynamics: Gyration on fully-packed loops

Apply the local move.



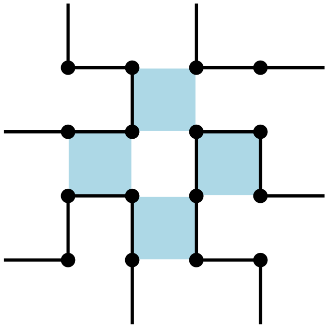
Dynamics: Gyration on fully-packed loops

Apply the local move.



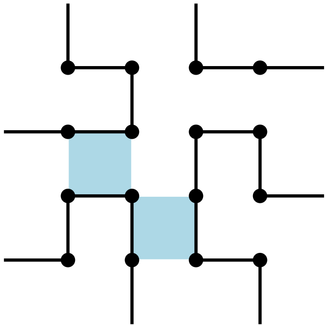
Dynamics: Gyration on fully-packed loops

Now consider the odd squares.



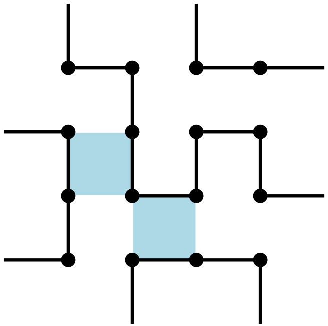
Dynamics: Gyration on fully-packed loops

Apply the local move.



Dynamics: Gyration on fully-packed loops

Apply the local move.



Dynamics: Gyration on fully-packed loops

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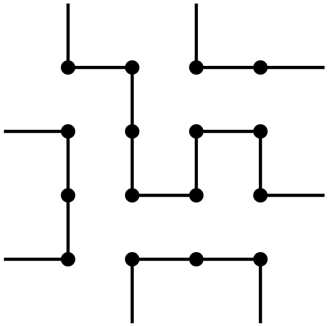

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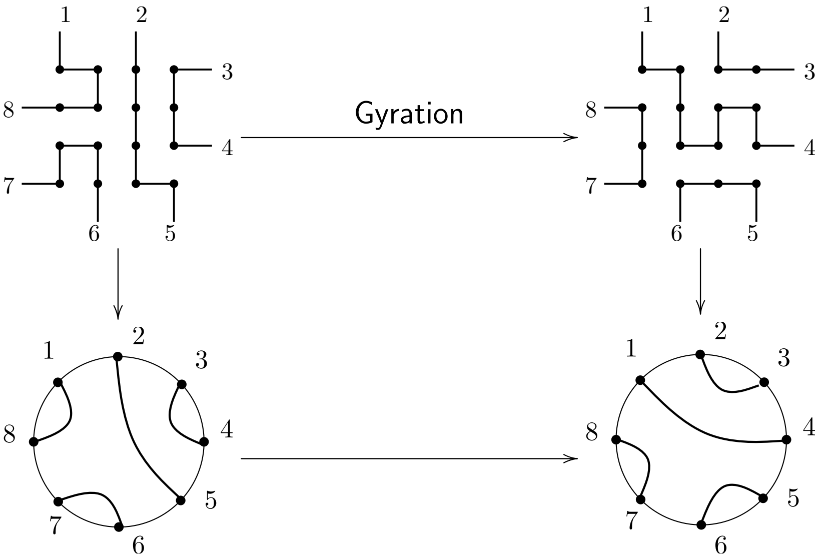
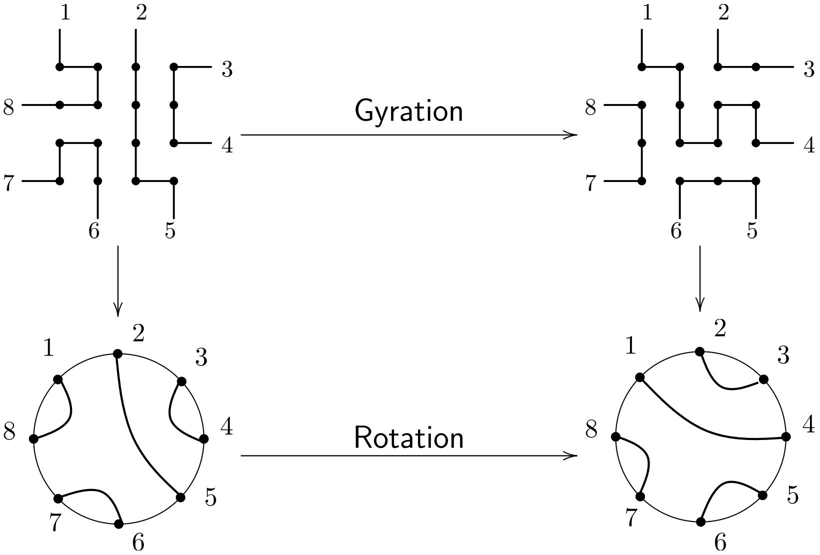


Dynamics: Gyration on fully-packed loops



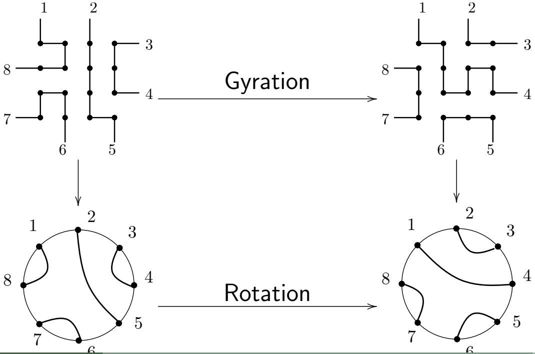
The square is a circle

Theorem (Wieland 2000)

Gyration on an $n \times n$ fully-packed loop rotates the **link pattern** *by an angle of $2\pi/2n$.*

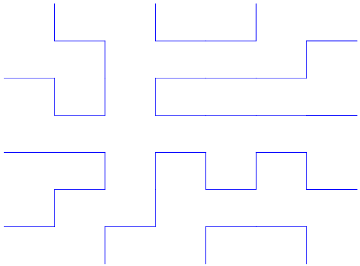
Corollary (Dilks, Pechenik, and S. 2017)

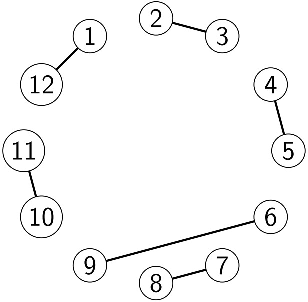
Gyration exhibits **resonance** *with frequency $2n$.*



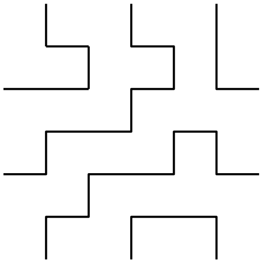
A 6 × 6 ASM with gyration orbit of length 84

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$





Dynamics: Gyration on the alternating sign matrix poset



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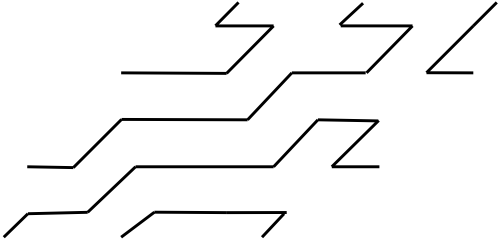
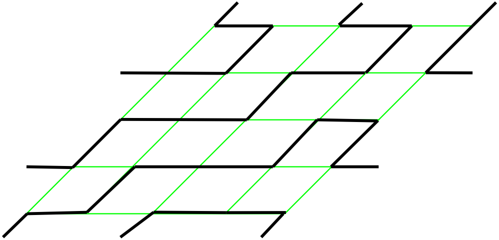


Dynamics: Gyration on the alternating sign matrix poset



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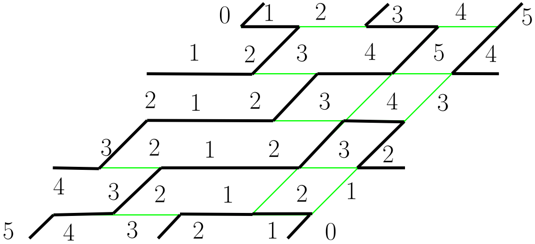

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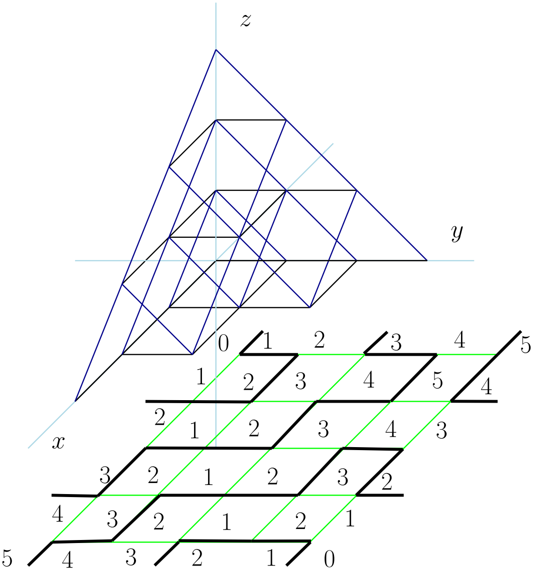

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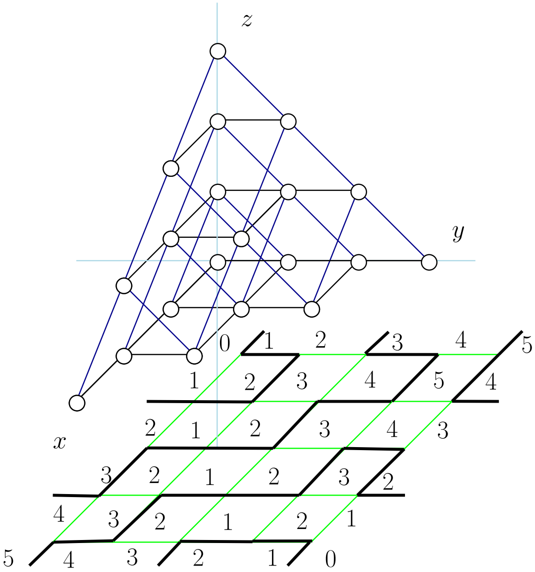

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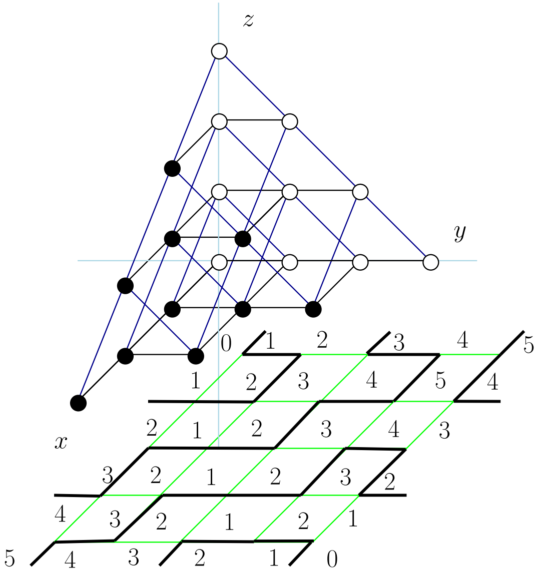

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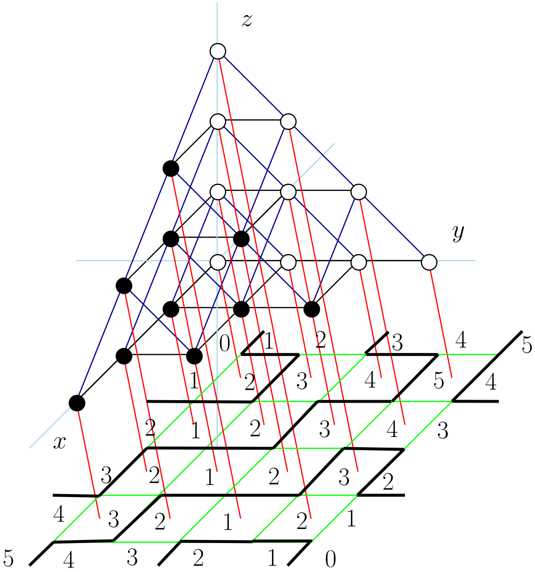

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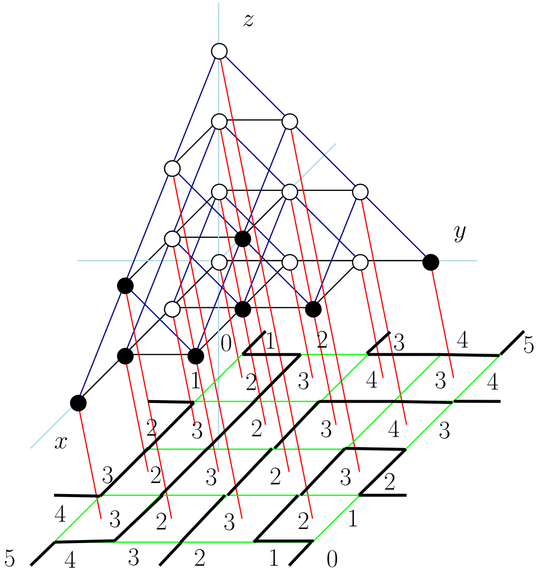

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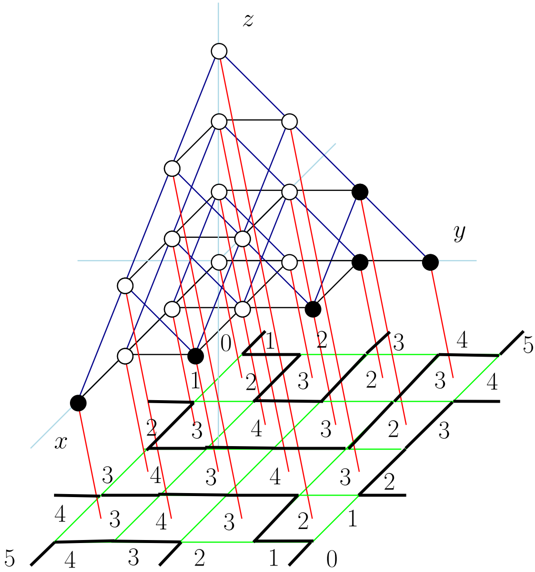
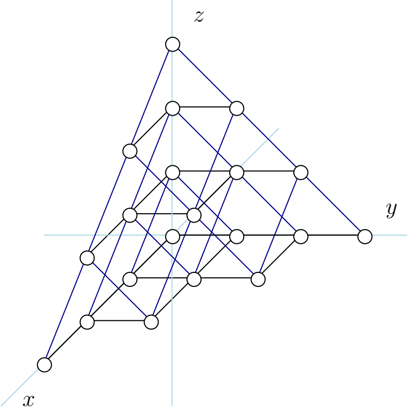


Dynamics: Gyration on the alternating sign matrix poset

Theorem (S. and Williams 2012)

Gyration on fully-packed loops is equivalent to **toggling** *even then odd ranks in the ASM tetrahedral poset.*



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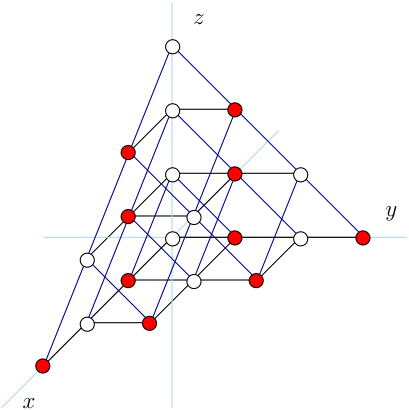

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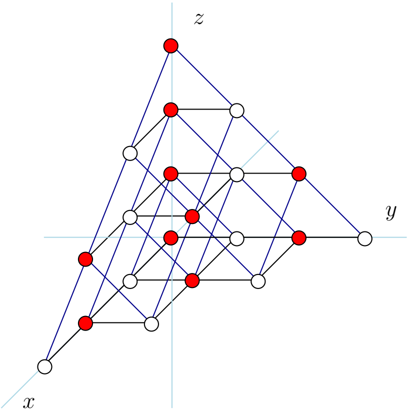


Toggles

For each $x \in P$ and each $I \in \mathcal{L} \subseteq 2^E$, the toggle $t_x(I)$ is the symmetric difference of I and $\{x\}$ provided the result is in $\mathcal{L}$ and the identity otherwise.

Definition (Cameron and Fon der Flaass 1995 (order ideals); S. 2018)

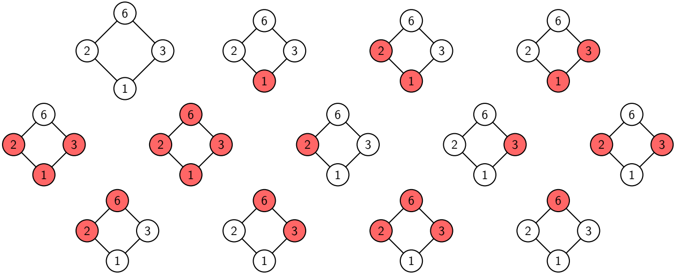
The *toggle group* $T(\mathcal{L})$ is the subgroup of the symmetric group $\mathfrak{S}_{\mathcal{L}}$ generated by the toggles $\{t_x\}_{x \in E}$.

Interesting toggle groups

Toggles have been studied in the following sets of subsets:

- Order ideals of a poset (many papers)
- Noncrossing partitions (Einstein, Farber, Gunawan, Joseph, Macauley, Propp, Rubinstein-Salzedo 2016)
- Independent sets of a graph (Joseph and Roby 2017)
- Antichains of a poset (Joseph 2019, Joseph and Roby 2020)
- Interval-closed sets of a poset (Elder, Lafrenière, McNicholas, S., Welch 2023, and current work LMSW + Lewis)

Toggles on interval-closed sets



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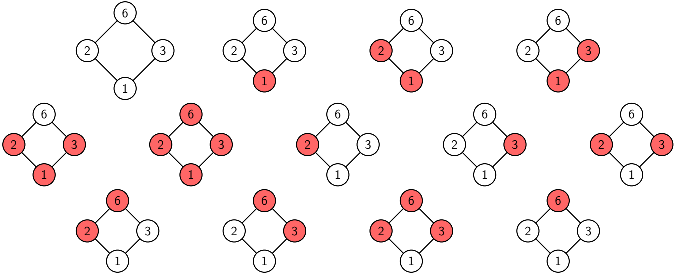
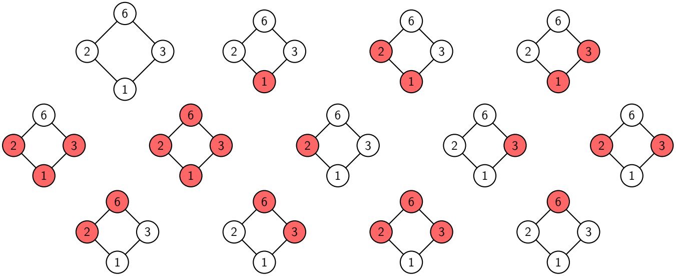


Example toggle actions:

Toggles on interval-closed sets



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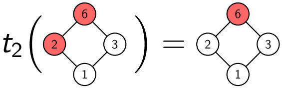

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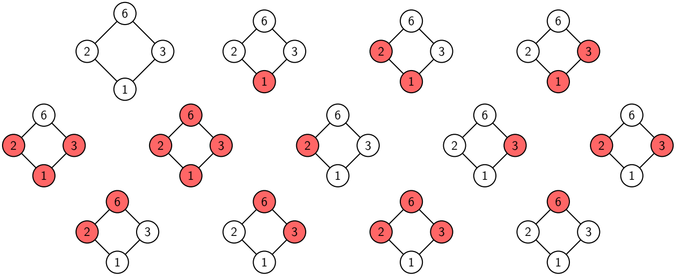

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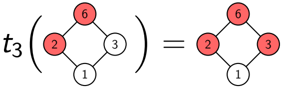

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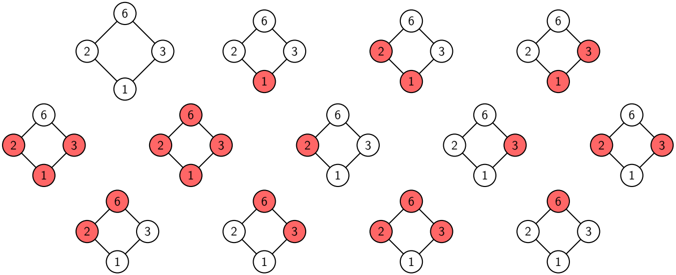

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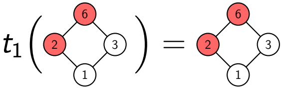
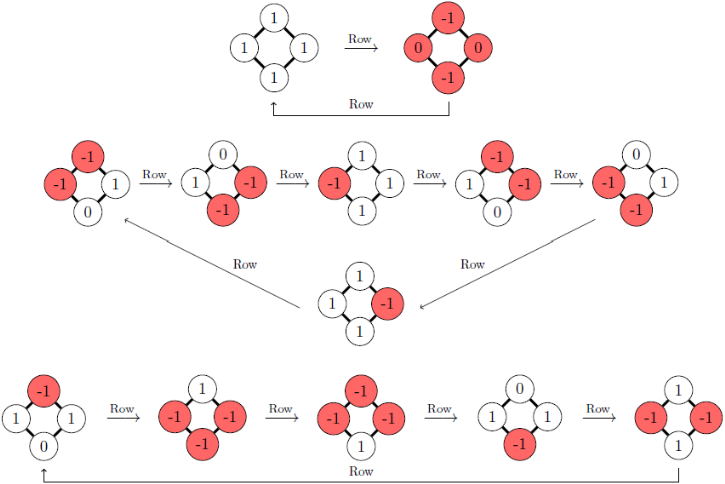


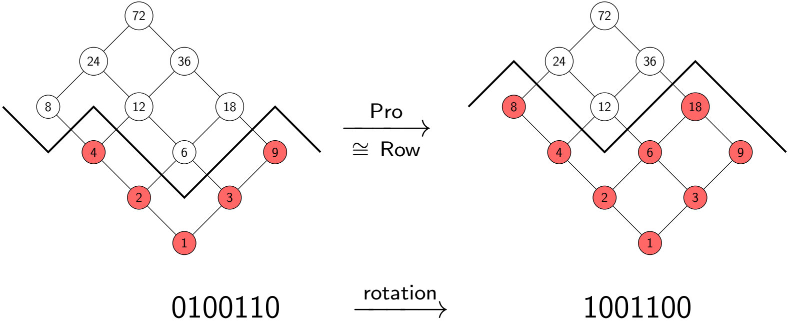
Rowmotion

Rowmotion is the toggle group action that composes all the toggles from top to bottom.



“Classical” (order ideal) rowmotion on products of chains

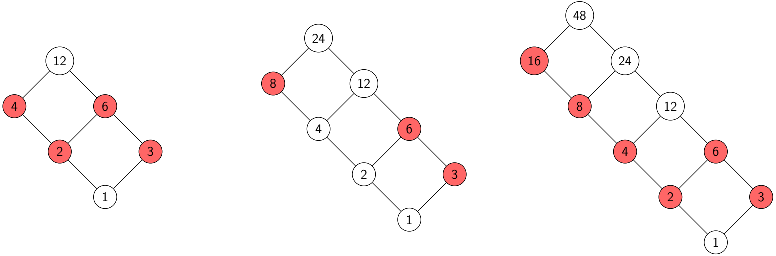
Order-ideal rowmotion on products of two chains $\mathbf{a} \times \mathbf{b}$ has order $a + b$. (S.-Williams 2012) used the toggle group to show it has the same orbit structure as toggling left-to-right, which is equivalent to rotation of a binary word of length $a + b$.



Interval-closed set rowmotion on products of chains

Problem (Open)

What is the order of rowmotion on interval-closed sets of $\mathbf{a} \times \mathbf{b}$*? The orbit structure?*



Order	60	210	1144
Orbits	$[2^2, 3^1, 6^2, 20^1]$	$[2^1, 5^3, 6^1, 7^4, 10^2]$	$[2^1, 4^2, 8^7, 26^1, 44^1]$

The conjugacy of rowmotion, promotion, and gyration

Theorem (S. and Williams 2012)

In any ranked poset, there are equivariant bijections between the order ideals under under rowmotion (toggle top to bottom), promotion (toggle left to right), and gyration (toggle evens then odds).

Rowmotion, promotion, and gyration
all have the same orbit structure!

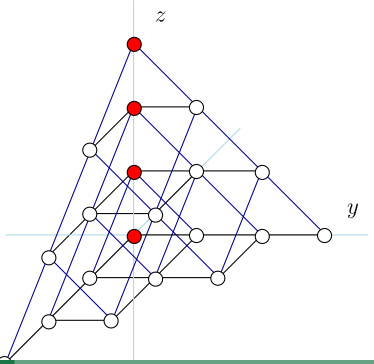
Gyration as a toggle group action

Theorem (S. and Williams 2012)

Gyration on fully-packed loops has the same orbit structure as rowmotion on order ideals of the ASM poset.

Corollary

Rowmotion on the ASM poset exhibits resonance with frequency $2n$.



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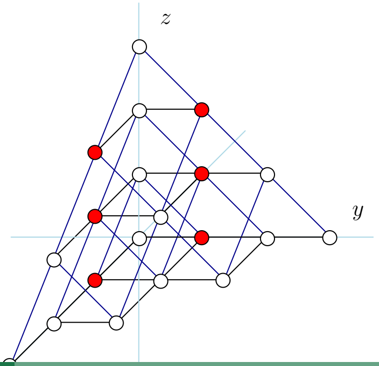

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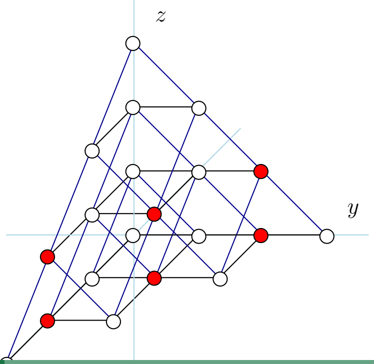

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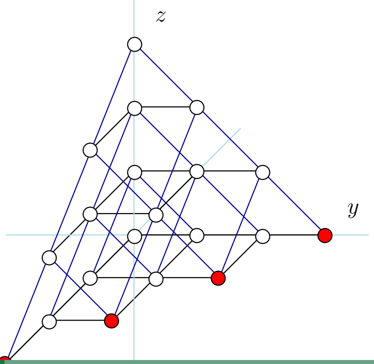

Homomesy

Homomesy in products of two chains

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Mathematics Subject Classifications: 05E18, 06A11

Abstract

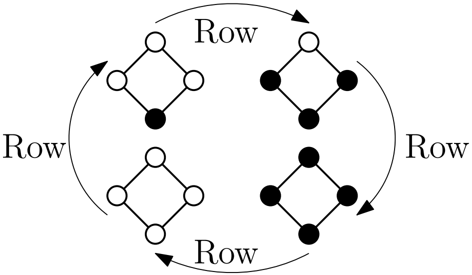
Many invertible actions τ on a set $\mathcal{S}$ of combinatorial objects, along with a natural statistic f on $\mathcal{S}$, exhibit the following property which we dub **homomesy**: the average of f over each τ-orbit in $\mathcal{S}$ is the same as the average of f over the whole set $\mathcal{S}$. This phenomenon was first noticed by Panyushev in 2007 in the context of the rowmotion action on the set of antichains of a root poset; Armstrong, Stump, and Thomas proved Panyushev's conjecture in 2011. We describe a theoretical framework for results of this kind that applies more broadly, giving examples in a variety of contexts. These include linear actions on vector spaces, sandpile dynamics, Suter's action on certain subposets of Young's Lattice, Lyness 5-cycles, promotion of rectangular semi-standard Young tableaux, and the rowmotion and promotion actions on certain posets. We give a detailed description of the latter situation for products of two chains.

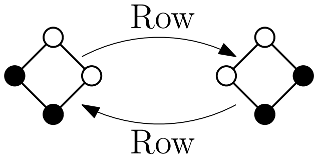
Keywords: antichains, ballot theorems, homomesy, Lyness 5-cycle, orbit, order ideals, Panyushev complementation, permutations, poset, product of chains, promotion, rowmotion, sandpile, Suter's symmetry, toggle group, Young's Lattice, Young tableaux.

The homomesy phenomenon

Definition (Propp and Roby 2015)

A statistic on a set exhibits *homomesy* with respect to an action when the orbit-average of the statistic equals the global average of that statistic.





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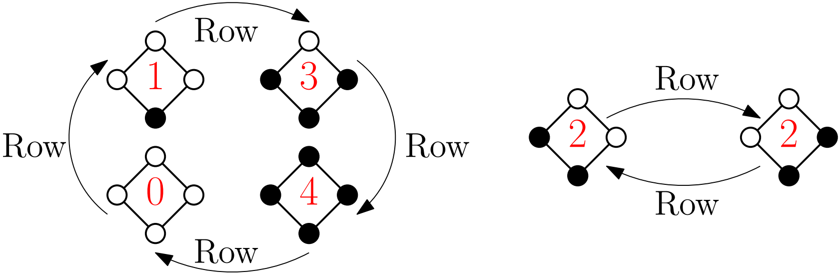
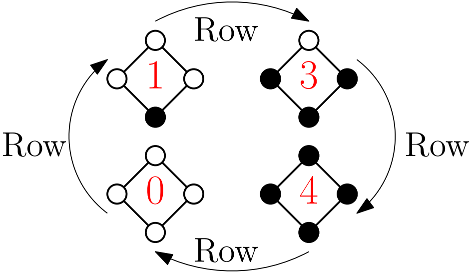


$$\frac{0+1+3+4}{4} = 2 \qquad \frac{2+2}{2} = 2$$

Homomesy of rowmotion in $\mathbf{a} \times \mathbf{b}$

Theorem (Propp and Roby 2015)

The cardinality statistic on order ideals of $\mathbf{a} \times \mathbf{b}$ *exhibits homomesy with respect to rowmotion.*



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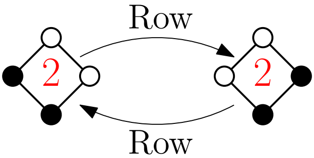
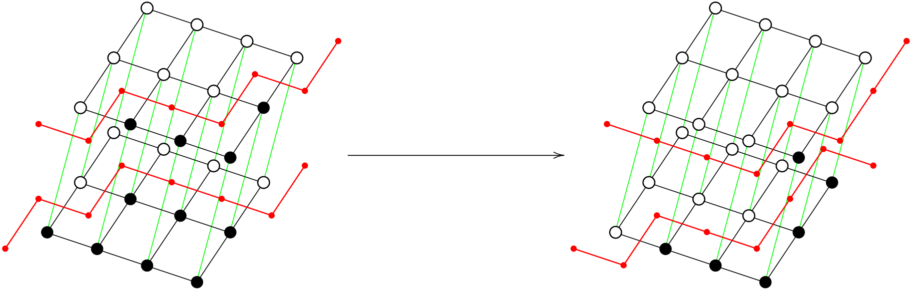


$$\frac{2+2}{2} = 2$$

Homomesy of rowmotion in $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$

Theorem (C. Vorland 2019)

The cardinality statistic on order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ exhibits homomesy with respect to rowmotion.



Homomesies on permutations - an analysis of the maps and statistics in the FindStat database [ELMSW 2024]

- One object: permutations
- Multiple actions and statistics!
- Goals:
  - characterize homomesy
  - understand the maps that exhibit a lot of homomesy.

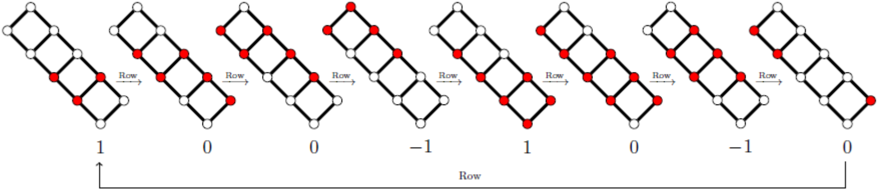
    $\Longrightarrow$ Systematic study.
- Outcomes:
  - 120+ proven occurrences of homomesy
  - Counter-examples for all other pairs of a map and a statistic from FindStat (7,000 +)

Toggling, rowmotion and homomesy on interval-closed sets [ELMSW 2023]

Define the max minus min statistic to be the number of maximal elements of I minus the number of minimal elements of I.

Theorem (ELMSW 2023)

The max minus min statistic is 0*-mesic for rowmotion on interval-closed sets of* $\mathbf{a} \times \mathbf{2}$.



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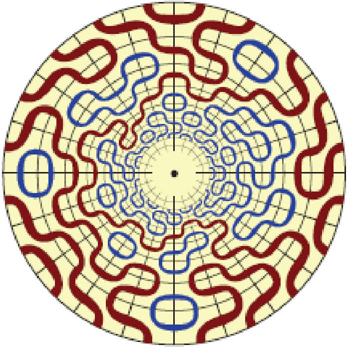
Conjecture

The max minus min statistic is 0-mesic for rowmotion on interval-closed sets of $\mathbf{a} \times \mathbf{b}$.

- Tested for all $a + b \leq 12$
- Does not hold for $\mathbf{2} \times \mathbf{2} \times \mathbf{5}$ or $\mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2}$

Razumov-Stroganov correspondence

$O(1)$ dense loop model on a semi-infinite cylinder



http://old-lipn.univ-paris13.fr/journee_calin/Slides/sportiello.pdf

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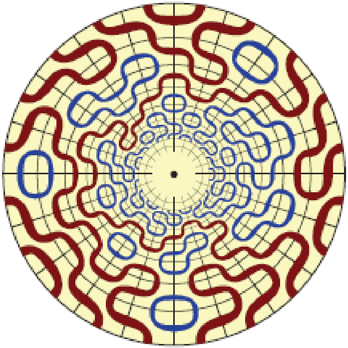

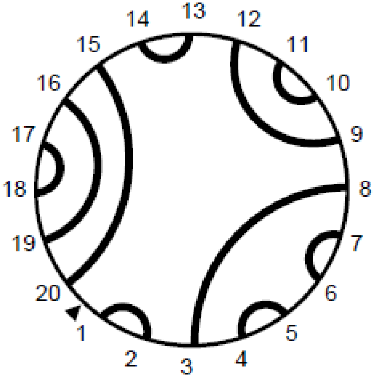

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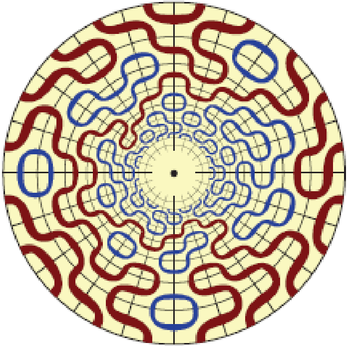
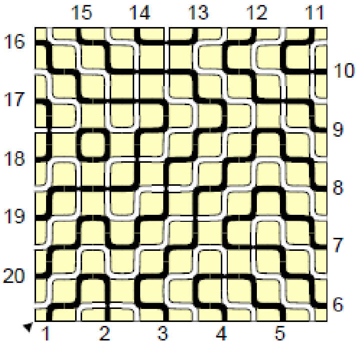


Fully-packed loop model



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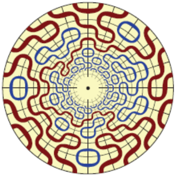
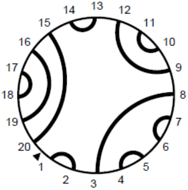
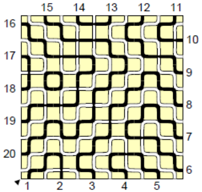
Gyration was used to prove...

Conjecture (Razumov and Stroganov 2004)

The probability that a configuration of the $O(1)$ dense loop model on a semi-infinite cylinder of perimeter $2n$ has link pattern π equals the probability that a fully-packed loop of order n has link pattern π.







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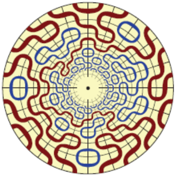

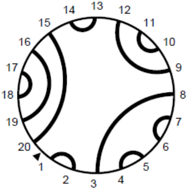

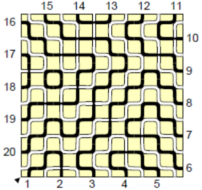

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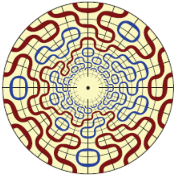
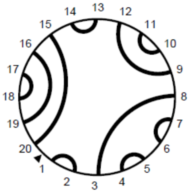
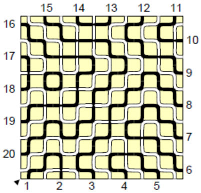






Lemma (Cantini and Sportiello 2011)

For a given square α, there are the same number of fully-packed loops in an orbit of gyration with configuration $|\alpha|$ as there are with configuration $\overline{\alpha}$.

The toggle group, homomesy, and the Razumov-Stroganov correspondence [S. 2015]

Definition (S. 2015)

Fix a poset P. For each $e \in P$, define the *toggleability* statistic $\mathfrak{T}_e$ as:

$$\mathfrak{T}_e(X) = \begin{cases} 1 & \text{if } e \text{ can be toggled } \textit{out} \text{ of } X, \\ -1 & \text{if } e \text{ can be toggled } \textit{in} \text{ to } X, \\ 0 & \text{otherwise.} \end{cases}$$

Wanted to prove:
For the ASM poset P and $e \in P$, $\mathfrak{T}_e$ is *homomesic* with average value 0 with respect to gyration.

The toggle group, homomesy, and the Razumov-Stroganov correspondence [S. 2015]

Definition (S. 2015)

Fix a poset P. For each $e \in P$, define the *toggleability* statistic $\mathfrak{T}_e$ as:

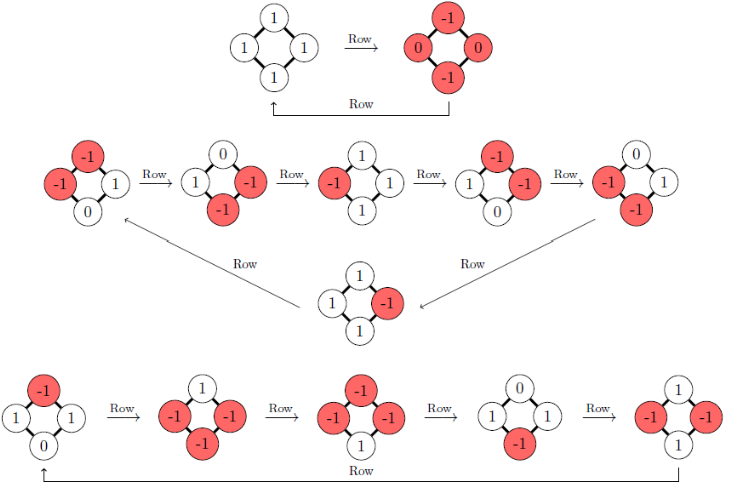
$$\mathfrak{T}_e(X) = \begin{cases} 1 & \text{if } e \text{ can be toggled } \textit{out} \text{ of } X, \\ -1 & \text{if } e \text{ can be toggled } \textit{in} \text{ to } X, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (S. 2015)

Given any ranked poset P and $e \in P$, $\mathfrak{T}_e$ is homomesic with average value 0 with respect to gyration.

Theorem (Elder, Lafreniere, McNicholas, S., Welch 2023)

For any poset, the toggleability statistic of any maximal or minimal element is 0-mesic under interval-closed set rowmotion.



HOMOMESY VIA TOGGLEABILITY STATISTICS

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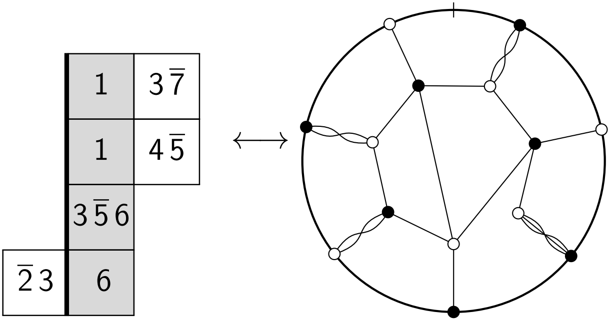
Abstract. The rowmotion operator acting on the set of order ideals of a finite poset has been the focus of a significant amount of recent research. One of the major goals has been to exhibit homomesies: statistics that have the same average along every orbit of the action. We systematize a technique for proving that various statistics of interest are homomesic by writing these statistics as linear combinations of "toggleability statistics" (originally introduced by Striker) plus a constant. We show that this technique recaptures most of the known homomesies for the posets on which rowmotion has been most studied. We also show that the technique continues to work in modified contexts. For instance, this technique also yields homomesies for the piecewise-linear and birational extensions of rowmotion; furthermore, we introduce a q-analogue of rowmotion and show that the technique yields homomesies for "q-rowmotion" as well.

SL_4 webs - dynamics showed the way

Theorem (Gaetz, Pechenik, Pfannerer, S., Swanson 2023+)

- *Contracted, fully-reduced, top hourglass plabic graphs with $4d$ boundary vertices give a rotation-invariant basis for SL_4.*
- *Promotion on rectangular 4 row fluctuating tableaux $\leftrightarrow$ rotation of hourglass plabic graph equivalence classes.*



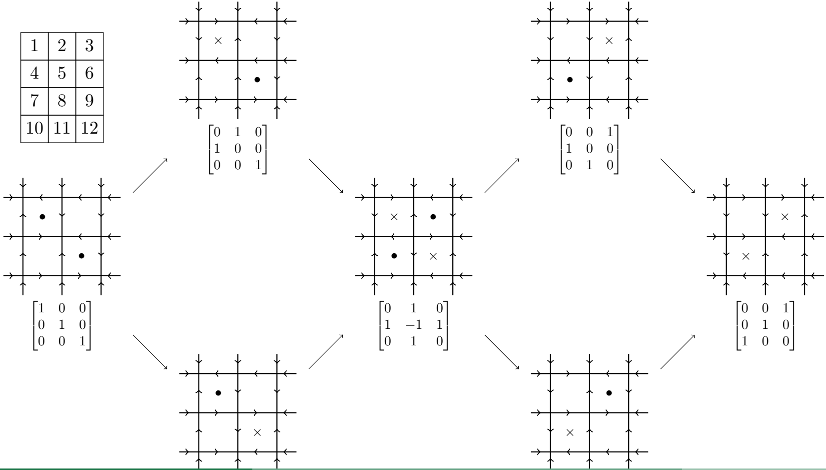
Talk: `sites.google.com/view/jessicastriker/home/research`

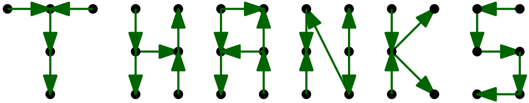
Alternating sign matrices as webs

Proposition (Gaetz, Pechenik, Pfannerer, S., Swanson 2023+)

The superstandard tableau of shape d^4 (lattice word $1^d 2^d 3^d 4^d$) is in bijection with the set of $d \times d$ alternating sign matrices.

1	2	3
4	5	6
7	8	9
10	11	12





Happy birthday Jim!