Lecture 3: Tropicalizations of Cluster Algebras – Examples

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Let A be a cluster algebra with B-matrix B. Let X be $\operatorname{Spec} A$ with all of the cluster variables inverted, and embed X into a torus by the cluster variables.

Every cluster variable is given by a Laurent polynomial which is also a positive rational function in every other cluster. So there are tons of opportunities to use the results on parametrization in the previous lecture.

Today, some examples. Next time, the work of Fock and Goncharov.

A note on the frozen variables

Let the B matrix be $n \times (m+n)$, so we have m frozen variables. Let $w = (w_1, w_2, \ldots, w_{m+n})$ be an integer vector with $wB = 0$. Then there is a $\mathbb{G}_m$ symmetry of X where u maps $(x_1, \ldots, x_n)$ to $(u^{w_1} x_1, \ldots, u^{w_n} x_n)$; this symmetry acts by a character on every cluster variable.

Example: Start with $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \\ \hline 1 & 1 \end{smallmatrix}\right)$, with initial cluster (x_1, x_2, y). So the two adjacent clusters are

$$\left(\frac{x_2 + y}{x_1}, x_2, y\right) \text{ and } \left(x_1, \frac{1 + x_1 y}{x_2}, y\right).$$

Take $w = (1, -1, -1)$. So $(x_1, x_2, y) \mapsto (ux_1, u^{-1}x_2, u^{-1}y)$. Notice that the new cluster variables $(x_2 + y)/x_1$ and $(1 + x_1 y)/x_2$ are homogenous for this action.

The corresponding statement about $\operatorname{Trop} X$ is that $\operatorname{Trop} X$ has a translation symmetry $(x_1, \ldots, x_n) \mapsto (x_1 + tw_1, \ldots, x_n + tw_n)$. I'll write $\operatorname{Trop}^0 X$ for the quotient of $\operatorname{Trop} X$ by all its translation symmetries. Note that $\dim Trop^0 X = (m+n) - \operatorname{Rank}(B)$.

Lemma If B has rank n, and B' is formed by adding one more row to B, then $\operatorname{Trop}^0 X \cong \operatorname{Trop}^0 X'$.

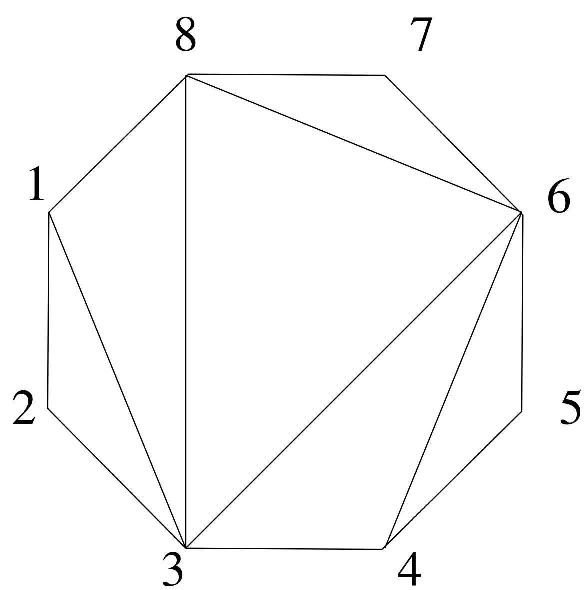
Corollary: If B and B' are two B-matrices with the same principal part, both of which have rank n, then
$\operatorname{Trop}^0 X \cong \operatorname{Trop}^0 X'$.

In this talk, all our examples have full rank. So $\operatorname{Trop}^0 X$ only sees the principal part of the B-matrix. In Fock and Goncharov's terminology, that means we don't have to distinguish between the $\mathcal{A}$-variety and the $\mathcal{X}$-variety.

Example 1: The Grasmannian $G(2, N)$

We have $\binom{N}{2}$ cluster variables p_{ij} for $1 \leq i < j \leq n = N$. The variables p_{12}, p_{23}, $\ldots$, $p_{(N-1)N}$, p_{N1} are frozen; the others are not.

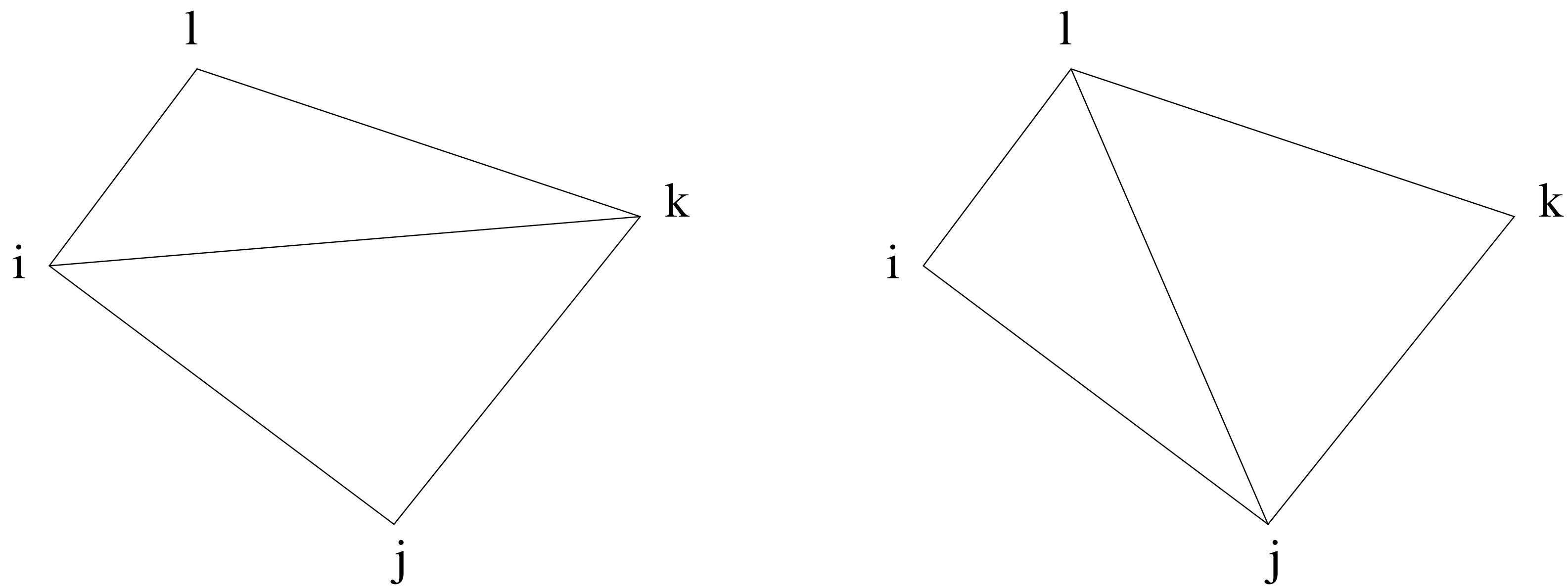
The clusters correspond to triangulations of the N-gon. In particular, the rank is $N - 3$ – the number of diagonals in every triangulation.



p_{13} p_{38} p_{68} p_{36} p_{46} p_{12} p_{23} p_{34} p_{45} p_{56} p_{67} p_{78} p_{18}

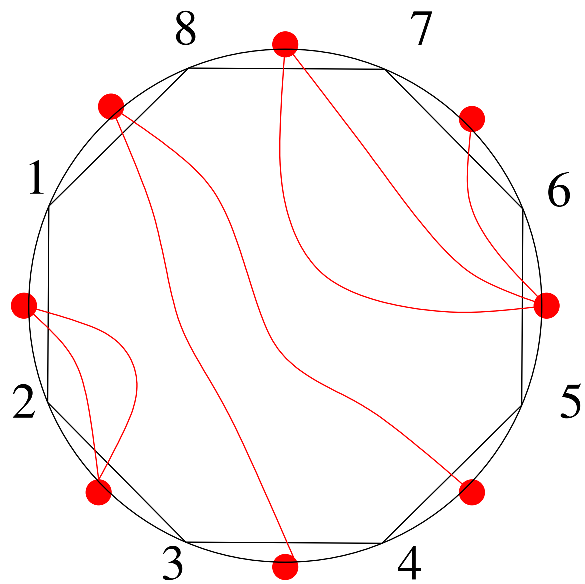
The exchange relations are

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk} \quad i < j < k < \ell.$$



$\mathrm{Trop}_0^+(X)$ is a simplicial fan of dimension $N-3$; it is combinatorially the dual fan of the associahedron. Choosing different initial clusters gives different polytopal realizations of the associahedron. Notice that the combinatorics of the fan reflects the cluster structure – this is still a mystery.

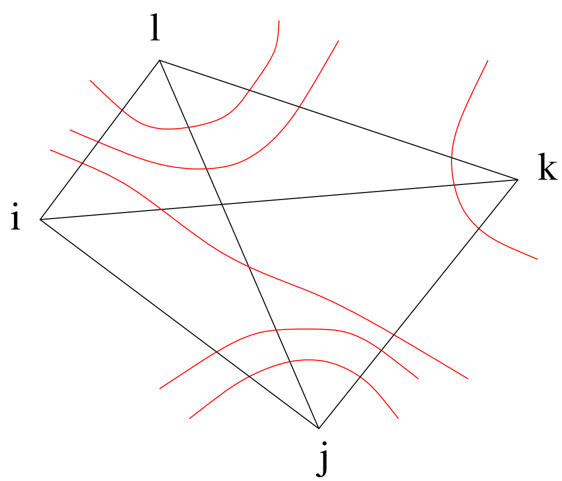
You can prove this by brute force (S-Williams) but the right proof is to think about laminations (W. Thurston, see also Fock-Goncharov and Fomin-Shapiro-D. Thurston).



A lamination is a collection of noncrossing arcs between the red dots. Given a lamination, set p_{ij} to be the number of red arcs crossed by the chord ij.

This gives a bijection between laminations and solutions to

$$p_{ik} + p_{j\ell} = \max(p_{ij} + p_{k\ell}, p_{i\ell} + p_{jk}).$$



$p_{ik} = 2 \quad p_{ij} = 2 \quad p_{i\ell} = 3$

$p_{j\ell} = 5 \quad p_{k\ell} = 3 \quad p_{jk} = 4$

Maximal cones in $\mathrm{Trop}^0_+ G(2,n)$ correspond to triangulations of the red N-gon; smaller cones correspond to partial triangulations.

Fine Print: Laminations describe solutions where the p_{ij} are ≥ 0 and where $p_{ik} + p_{j\ell} \equiv p_{ij} + p_{k\ell} \equiv p_{i\ell} + p_{jk} \mod 2$. In general, we need to allow real weights, and to allow negative multiplicities at the corners.

Example 2: Rank 2, infinite type

Let $B = \left(\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right)$. There is a bi-infinite sequence of cluster variables x_m, obeying

$$x_{m+1}x_{m-1} = x_m^2 + 1.$$

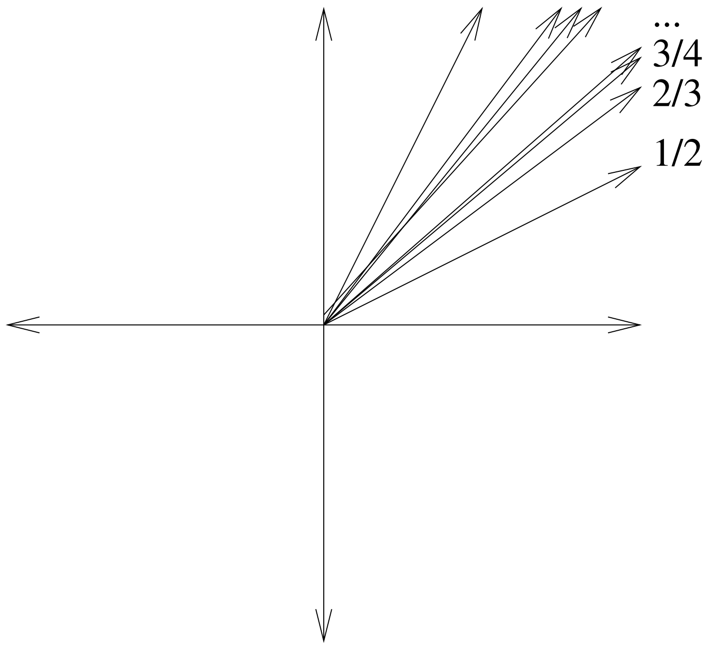
Tropically,

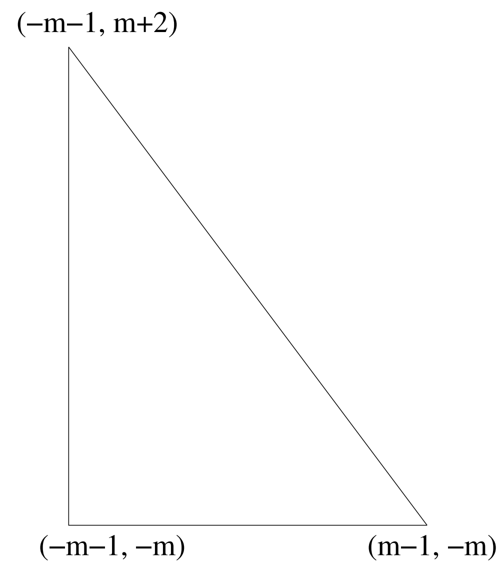
$$x_{m+1} = \max(2x_m, 0) - x_{m-1}.$$

Solutions: Either all of the x_m's are equal and positive, or we have x_k and $x_{k+1} \leq 0$ and the sequence is linear and positive for m on either side of k.

$$3.7 \quad 2.7 \quad 1.7 \quad 0.7 \quad -0.3 \quad -0.7 \quad 0.3 \quad 1.3 \quad 2.3$$

The tropical fan has lines in direction $(-1, 0)$, $(0, -1)$, $(m, m+1)$ and $(m+1, m)$. I've labeled some of the rays with their slopes.

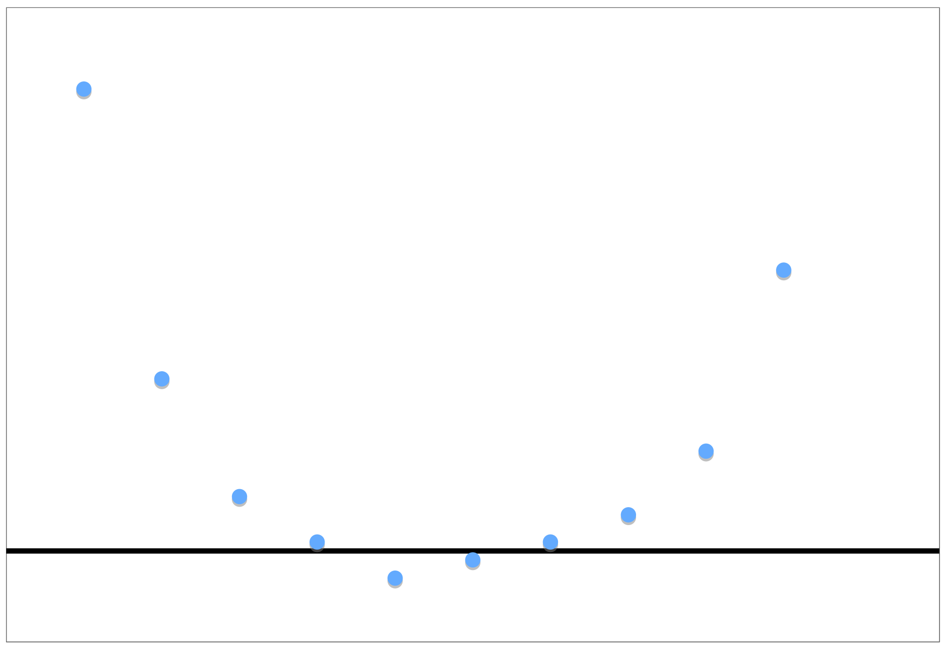




Sherman-Zelevinsky compute the corresponding Newton polytopes. They are triangles. For m positive, the vertices are at $(-m-1, -m)$, $(-m-1, m+2)$ and $(m-1, -m)$; there is a similar formula for $m < 0$.

Now let $B = \left(\begin{smallmatrix} 0 & 3 \\ -3 & 0 \end{smallmatrix}\right)$. The picture is the same as before, but with exponential growth of the form $h_{n+1} = 3h_n - h_{n-1}$ instead of linear.

$$51 \quad 19 \quad 6 \quad 1 \quad -3 \quad -1 \quad 1 \quad 4 \quad 11 \quad 31$$



The Newton polytopes are triangles $(-G_{m+1}, -G_m)$, $(-G_{m+1}, G_{m+2})$ and $(G_{m-1}, -G_m)$ where G_m are the even Fibonacci's 1, 3, 8, 21, 55, ...

The fan this time has a positive area cone corresponding to solutions to $x_{m+1} = 3x_m - x_{m-1}$ with all the $x_i > 0$.