Tropicalizations of Positive Parts of Cluster Algebras

The conjectures of Fock and Goncharov

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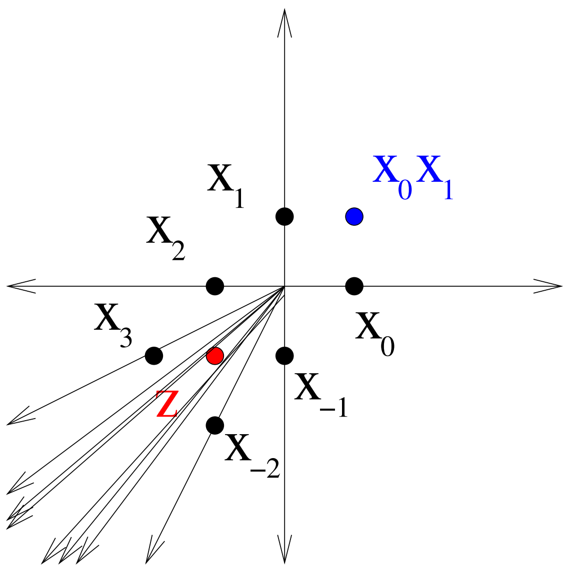
This talk is based on `arXiv:math/0311245`, section 4.

We saw before that tropicalizations look like cluster complexes. We would like to make this precise.

Idea: There should be a map from lattice points in $\mathrm{Trop}_+ X$ to the cluster algebra A.

Simplifications: B skew-symmetric and the rows of B span $\mathbb{Z}^n$ over the integers. (Last condition, as opposed to merely requiring full rank, added thanks to a correction by Bernhard Keller.)

Apology for signs: My sign conventions and FG's are incompatible. Some pictures which you saw on previous slides will reappear rotated in order to match the new conventions.



Recall from last time the recursion $x_{m+1}x_{m-1} = x_m^2 + 1$. A is the sub-algebra of $k(x_1, x_2)$ generated by all the x_i. We want to send lattice points in this picture to elements of A.

The black dots are sent to the cluster variables. Multiples of these points are sent to powers of the cluster variables.

The lattice points inside the cones are sent to corresponding monomials. We show the point sent to $x_0 x_1$.

The element z is $\det\left(\begin{smallmatrix} x_0 & x_1 \\ x_2 & x_3 \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} x_2 & x_3 \\ x_4 & x_5 \end{smallmatrix}\right) = \cdots$. The powers of this lattice point are sent to the Chebyshev polynomials $z^2 - 2$, $z^3 - 3z$, $z^4 - 4z^2 + 2$, $\ldots$

For every cluster I, we have a piecewise linear convex parameterization $\phi_I : \mathbb{R}^I \to \mathrm{Trop}^+(X)$ (discussed last time). The images of $\mathbb{Z}^I$ all match up – call these $\mathrm{Trop}^+_{\mathbb{Z}}(X)$. (FG's notation is $\mathcal{A}(\mathbb{Z}^t)$.)

Fock and Goncharov conjecture the existence of a map $\mathbb{I} : \mathrm{Trop}^+_{\mathbb{Z}}(X) \to A$ with four good properties.

1. Parametrization of the cluster complex
2. Symmetry
3. Positivity
4. Leading terms

Warning: My numbering 1,2,3,4 is FG's 1,3,4,2.

Condition 1: Parametrization of the cluster complex

The composite $\mathbb{Z}^I \xrightarrow{\phi_I} \mathrm{Trop}^+_{\mathbb{Z}}(X) \xrightarrow{\mathbb{I}} A$ sends $\mathbb{Z}^I_{\geq 0}$ to the monomials in the cluster variables $\{x_i\}_{i \in I}$.

So all of the cluster monomials appear in the image of $\mathbb{I}$.

The preimage of the cluster monomials is supposed to be the g-vector fan at seed I. I haven't seen this written down, but I believe experts know it.

Consequences for denominators

Let J_1 and J_2 be two clusters containing x_j. Then the vector e_j in $\mathbb{Z}^{J_1}$ and $\mathbb{Z}^{J_2}$ must have the same image. This implies that, for $i \in J_1$, if we write x_i as a Laurent polynomial in $\{x_{j'}\}_{j' \in J_2}$, then x_j does not appear in the denominator.

This would follow if there were a sequence of mutations taking J_1 to J_2 all of whose intermediate elements contain x_j. This is still an open conjecture.

More generally, let i be any cluster variable. Write $x_i = F(x_{j_1}, \ldots, x_{j_n}) = F'(x_{j'_1}, \ldots, x_{j'_n})$. Then F and F', tropically evaluated at e_j, should give the same result. So the power of x_j occurring in the denominator of F is the same as the power occurring in the denominator of F'. Call this $d(i, j)$.

So Condition 1 implies that there is a well-defined function $d(i, j)$ on pairs of cluster variables as above. In finite type, this is the compatibility degree from *Y-systems and generalized associahedra.* For triangulations of the N-gon, or of surfaces, $d(i, j)$ is 1 if chords i and j cross and zero otherwise.

Is this known in general?

Condition 2: Symmetry

Let a and $b \in \mathrm{Trop}^{+}_{\mathbb{Z}}(X) \subset \mathrm{Trop}^{+}_{0}(X)$. Send a to a function $f = \mathbb{I}(a)$ in the cluster algebra. Then we have a piecewise linear functions $\mathrm{Trop}(f)$ on $\mathrm{Trop}^{+}_{0}(X)$. Explicitly, lift $b \in \mathrm{Trop}\, X$ to a point $x \in X \cap \mathcal{R}^{N}_{+}$; the function $\mathrm{Trop}(f)$ is $v(f(x))$.

Define $d(a,b) = \mathrm{Trop}(\mathbb{I}(a))(b)$. Assuming condition 1, when $\mathbb{I}(a) = x_i$ and $\mathbb{I}(b) = x_j$, this is the function $d(i,j)$ from the last slide.

The symmetry condition: $\boxed{\mathrm{d(a,b) = d(b,a)}}$

If we pull d back to $\mathbb{Z}^I$ along ϕ_I, then $d(a,b)$ is piecewise linear, convex and homogenous in b, because it is the tropicalization of a Laurent polynomial. So, assuming the symmetry conjecture, $d(a,b)$ is piecewise linear, convex and homogenous in both variables.

Consequences for Newton polytopes

Fix a cluster I. By condition 1, we are supposed to have $\mathbb{I}(\phi_I(\sum a_i e_i)) = \prod x_i^{a_i}$, for $a_i \geq 0$. So

$$\operatorname{Trop} \mathbb{I}(\phi_I(\sum a_i e_i)) = \sum a_i \operatorname{Trop} \mathbb{I}(\phi_I(e_i))$$

Assuming symmetry, this means that, for any $v \in \operatorname{Trop}^+_{\mathbb{Z}}(X)$, the function $d(v, \phi_I(\sum a_i e_i))$ should be linear in the a's, for $a \geq 0$. We know that it should be piecewise linear, but this condition tells us it is linear. So this is the condition that says that $\phi_I : \mathbb{Z}^I \to \operatorname{Trop}^+ X$ should be linear on $\operatorname{Span}_+ e_i$. Taking this statement for all clusters I, this explains why we see the cluster complex as domains of linearity for the parametrization.

Explicitly, when we write f as a Laurent polynomial in $\{x_i\}_{i \in I}$, its numerator should have nonzero constant term. When f is a cluster variable itself, this is a weaker version of *Cluster Algebras IV*, Conjecture 5.4, now proved by Derksen-Weyman-Zelevinsky.

More generally, every cone of the g-vector fan at seed I should be a subcone of the normal cone at some vertex of the Newton polytope of f. Again, for f a cluster variable, this follows from conjectures in *CA:IV*, now proved by DWZ.

Condition 3: Positivity

Let $A_+ \subset A$ be $\bigcap_I \mathbb{Z}_{\geq 0}[x_{i_1}^{\pm}, \ldots, x_{i_1}^{\pm}]$. It is not clear that there are any nonconstant elements in A_+.

If we assume the positivity conjecture for Laurent polynomials, then the cluster variables are in A_+.

The positivity condition: Every element of A_+ can be uniquely expressed as a positive linear combination of $\mathbb{I}(\mathrm{Trop}_{\mathbb{Z}}^{+}(X))$.

This is really three claims:

- $\mathbb{I}(\mathrm{Trop}_{\mathbb{Z}}^{+}(X)) \subset A_+$
- $\mathbb{I}(\mathrm{Trop}_{\mathbb{Z}}^{+}(X))$ positively spans A_+
- $\mathbb{I}(\mathrm{Trop}_{\mathbb{Z}}^{+}(X))$ are linearly independent in A.

I couldn't find this written down explicitly in FG, but it is also expected that $\mathbb{I}(\mathrm{Trop}_{\mathbb{Z}}^{+}(X))$ spans A.

Consequences for positivity

Clearly, A_+ is closed under multiplication. So, for f and $g \in \mathbb{I}(\mathrm{Trop}_{\mathbb{Z}}^{+}(X))$ we have $fg \in \mathbb{R}_{\geq 0} \cdot \mathbb{I}(\mathrm{Trop}_{\mathbb{Z}}^{+}(X))$.

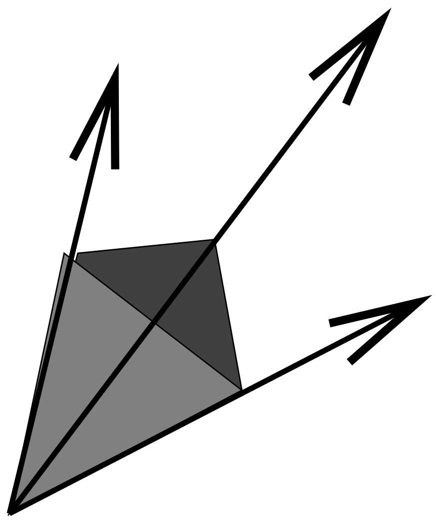
Conversely, suppose we have *any* subset $\mathcal{B}$ of A such that

- $\mathcal{B}$ is linearly independent in A
- $\mathcal{B}$ contains the cluster monomials and
- For f and g in $\mathcal{B}$, we have $fg \in \mathbb{R}_{\geq 0} \cdot \mathcal{B}$.

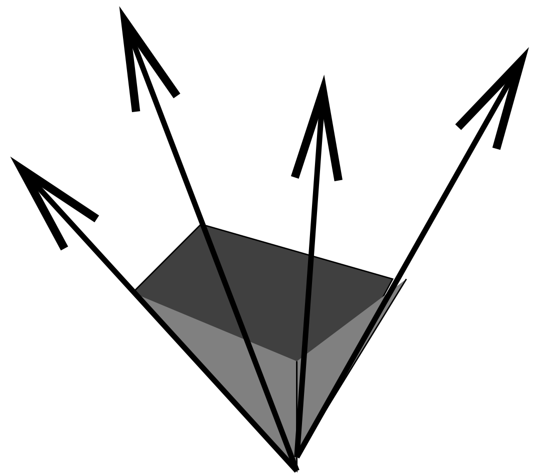
Such a set would necessarily lie in A_+. In particular, the existence of such a set implies positivity of Laurent coefficients. Fomin and Zelevinsky would love to know *any* such $\mathcal{B}$.

It is easy to show that there is at most one linearly independent subset $\mathcal{B}$ of A_+ which positively spans A_+.

Finite dimensional analogy: A cone in $\mathbb{R}^n$ is the positive span of at most one linearly independent set of rays.



Positive span of a basis



Not the positive span of a basis

Condition 4: A condition on leading terms

I'm going to skip this one.

It says that a certain point must be vertex of the Newton polytope, with a dual cone that contains a certain cone.

Summary

We have map $\mathbb{I} : \mathrm{Trop}^+_{\mathbb{Z}}(X) \to A$ such that:

Parametrization of cluster complex: The composite $\mathbb{Z}^I \xrightarrow{\phi_I} \mathrm{Trop}^+_{\mathbb{Z}}(X) \xrightarrow{\mathbb{I}} A$ sends $\mathbb{Z}^I_{\geq 0}$ to the monomials in the cluster variables $\{x_i\}_{i \in I}$.

Symmetry: Define $d(a, b) = \mathrm{Trop}(\mathbb{I}(a))(b)$. Then $d(a, b) = d(b, a)$.

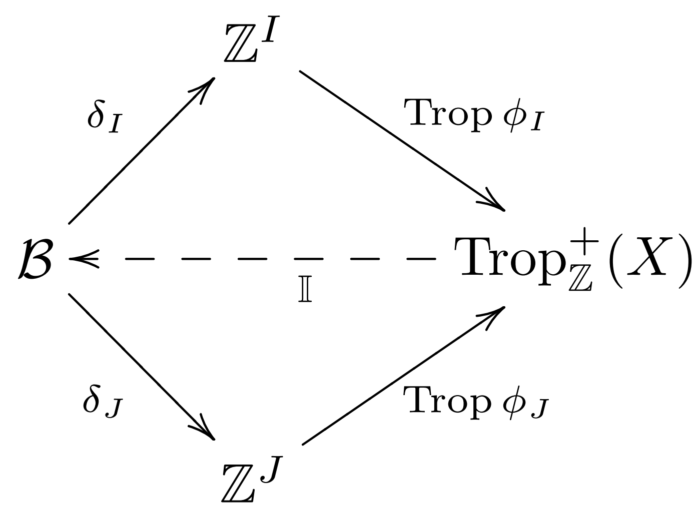
Positivity: Define $A_+ \subset A$ to be the elements which are positive Laurent polynomials in every cluster. Then every element of A_+ is uniquely a positive linear combination of $\mathbb{I}(\mathrm{Trop}^+_{\mathbb{Z}}(X))$.

A condition on leading terms: I skipped it.

Uniqueness of the FG construction

There is at most one set $\mathcal{B}$ of free positive generators for A_+. If there is no such set, the FG conjectures fail.

Fix a cluster I. Write f of $\mathcal{B}$ as $f_I \in \mathbb{Z}[x_i^{\pm}]_{i \in I}$. If the denominator vectors of the f's do not cover $\mathbb{Z}^I$ exactly once, the FG conjectures fail. If they do, we get $\delta_I : \mathcal{B} \to \mathbb{Z}^I$.



We want to fill in the diagram to a map $\mathbb{I} : \mathrm{Trop}^+_{\mathbb{Z}}(X) \to \mathcal{B}$.

Even if you get this far, the conjectures can still fail!

A change of topic

Let X be the double Bruhat cell $G^{w_0 w_0}$ in GL_n. Explicitly, this is $n \times n$ matrices none of whose upper left or lower right minors vanish. This has a cluster structure.

Parametrizations of $X(\mathbb{R}_+)$ One set of obvious coordinates on X are the matrix entries x_{ij}. Another set is to use the parametrization of X using double wiring diagrams. The change of coordinates formula between these coordinates is a positive rational map

$$W : \text{Wiring Coordinates} \to \mathbb{R}^{n^2}$$

.

Bases of functions on $n \times n$ matrices

The monomial basis There is an obvious basis of functions on $n \times n$ matrices: The monomials in the x_{ij}'s. These are indexed by $n \times n$ matrices with entries in $\mathbb{Z}_{\geq 0}^{n^2}$

The representation theory basis By Cauchy's identity, or by the Peter-Weyl theorem,

$$\mathcal{O}(\mathrm{Mat}_{n \times n}) \cong \bigoplus_{\lambda} V_\lambda \otimes V_\lambda^*.$$

So functions on $n \times n$ matrices have a basis indexed by ordered pairs of semistandard Young Tableaux of the same shape. Using Gelfand-Tsetlin patterns instead of SSYT's, we can index this basis by a certain cone $K \subset \mathbb{Z}^{n^2}$.

We have two different cones, $\mathbb{Z}_{\geq 0}^{n^2}$ and K, inside $\mathbb{Z}^{n^2}$. There is a classical piecewise linear bijection

$$jdt : \mathbb{Z}_{\geq 0}^{n^2} \to K$$

Theorem (Kirillov-Yamada, see also Danilov-Koshevoy) $\mathrm{Trop}\, W$, restricted to K, is jdt^{-1}.

Why am I raising this: $\mathrm{Trop}\, W$ is a map between Trop^+(Wiring Coordinates) and $\mathrm{Trop}^+ \mathbb{R}^{n^2}$. *Jeau de tacquin* is a bijection between two bases of functions on $n \times n$ matrices. Fock-Goncharov sends points of Trop^+ to bases of algebras.

Is there a commutative diagram?