Lecture 5: Tropical Geometry and Compactifications

David Speyer

This talk is based on Tevelev *Compactifications of Subvarieties of Torii*, and extensions thereof. They have been useful in many areas of tropical geometry.

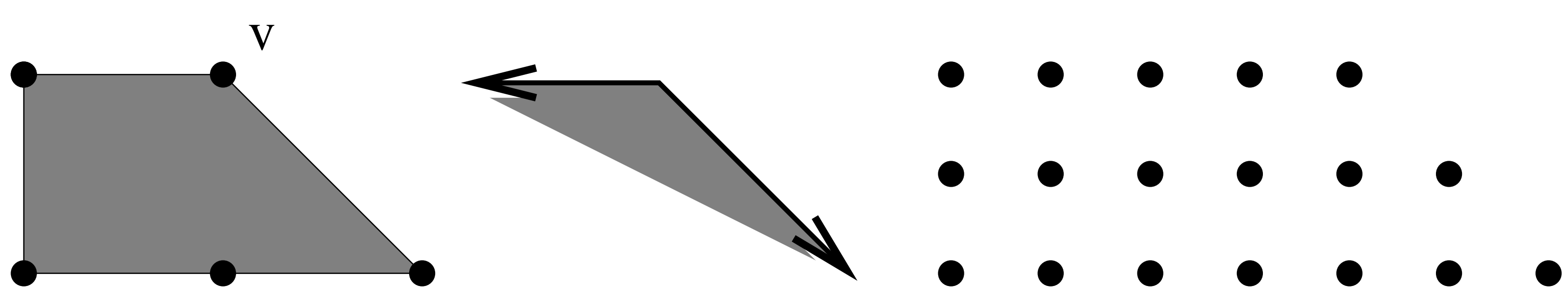
These ideas show up in Fock and Goncharov's 2011 preprint *Cluster X-varieties at infinity*, but other than that haven't had much tropical application yet. I think there should be a lot more to do here.

Polytopes and toric varieties

Let P be a lattice polytope in $\mathbb{R}^n$. For each vertex $v \in P$:

- Let T_v be the cone spanned by $u - v$ with $u \in P$.
- Let S_v be the semigroup $T_v \cap \mathbb{Z}^n$.
- Let $k[S_v]$ be the semigroup ring of S_v.
- Let $U_v = \operatorname{Spec} k[S_v]$.

The toric variety $\operatorname{Toric}(P)$ is made by gluing together the open affines U_v.



Toric varieties break up combinatorially into smaller toric varieties, indexed by the faces of the polytope.

Let $X \subset (\mathbb{C}^*)^n$ be a hypersurface with Newton polytope P. Let $\overline{X}$ be the closure of X in $\mathrm{Toric}(P)$.

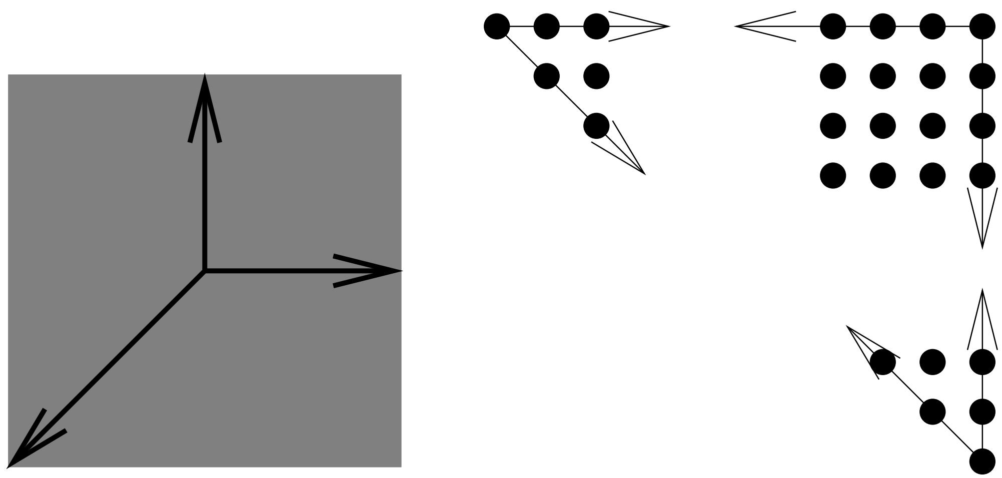
Then $\overline{X}$ meets the various faces of $\overline{X}$ in the varieties $\mathrm{in}_w X/(\mathrm{torus})$.

If w is dual to a face of dimension d, then the corresponding stratum of $\overline{X}$ is of dimension $d - 1$.

When all the $\mathrm{in}_w X$ are smooth, we say that X is "transverse to its Newton polytope". This condition comes up a lot in computational algebraic geometry.

Everything works the same if we replace P by some larger polytope whose normal fan refines that of P.

Things get much more interesting when we are dealing with non-hypersurfaces. For that, we need to present the description of toric varieties in terms of fans. This is more fundamental, but less intuitive.



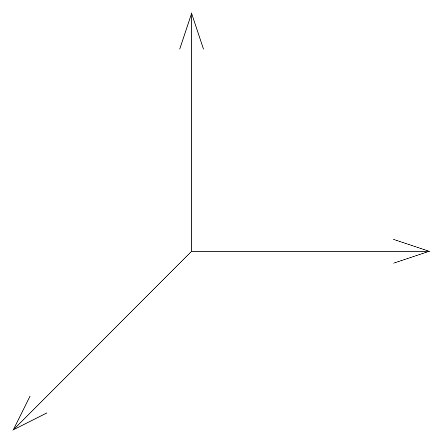
For a cone $\sigma \subset \mathbb{R}^n$, let $\sigma^\vee$ be the dual cone. Let $k[\sigma^\vee \cap \mathbb{Z}^n]$ be the semigroup ring of the lattice points in $\sigma^\vee$ and let U_σ be $\operatorname{Spec} k[\sigma^\vee \cap \mathbb{Z}^n]$.

The toric variety Toric(Σ) is glued from the U_σ.

If $\dim \sigma = m < n$, then $k[\sigma^\vee \cap \mathbb{Z}^n]$ has m nontrivial units.

Again, Toric(Σ) breaks into closed pieces V_σ indexed by the cones (not just the maximal cones) of Σ. This correspondence is containment reversing; $\dim V_\sigma = n - \dim \sigma$.

Deleting a cone σ deletes V_σ. So



is $\mathbb{P}^2 \setminus \{\text{three points}\}$.

Start with $X \subset (\mathbb{C}^*)^n$. We can take the closure $\overline{X}$ of X in $\mathrm{Toric}(\Sigma)$ for any fan Σ. Tevelev's realization is that a good choice of Σ is $\mathrm{Trop}\, X$.

Specifically, $\overline{X}$ is compact (proper) if and only if $\Sigma \supseteq \mathrm{Trop}\, X$.

If we have a maximal cone σ of Σ whose interior is disjoint from $\mathrm{Trop}\, X$, then $V_\sigma \cap \overline{X} = \emptyset$.

We can choose to put a fan structure Σ on $\mathrm{Trop}\, X$ such that

- For every $w \in \mathrm{Trop}\, X$, if σ is the cone whose relative interior w lies in, then $\overline{X} \cap V_\sigma \cong \mathrm{in}_w X/(\mathrm{torus})$.
- There is a flat family over $\mathrm{Toric}(\Sigma)$ whose fibers over V_σ are all isomorphic to $\mathrm{in}_w X$.

Tevelev calls $\overline{X}$ a "tropical compactification" of X.

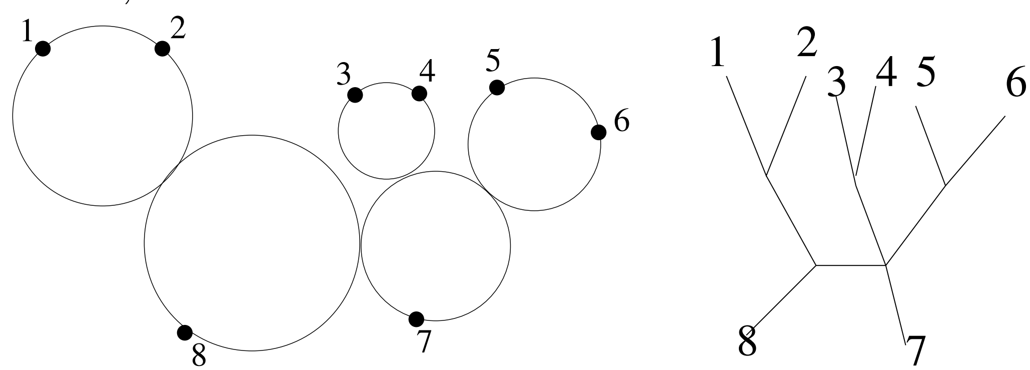
We can take Σ to be simplicial. If we do this, then $\overline{X}$ is "combinatorially normal crossings" meaning, roughly, that all the strata of X have the dimensions we expect them to. Specifically, for ρ a ray of Σ, let $D_\rho = \overline{X} \cap V_\rho$. Then D_ρ is codimension 1 (not necessarily irreducible). The intersection $D_{\rho_1} \cap D_{\rho_2} \cap \cdots \cap D_{\rho_k}$ is either empty or codimension k; specifically, it is nonempty if and only if the ρ_i span a cone in Σ.

A note on torus symmetry: Suppose that X has a nontrivial torus stabilizer H. Then $\operatorname{Trop} X$ has a translation symmetry by a vector space of dimension $\dim H$. The toric variety technology wants our fans to have cones with vertices, and we can't insert a vertex without breaking the translation symmetry. Tevelev compactifies X/H, not X. If we want to compactify X, we need to make an unnatural choice of (1) a splitting of $H \to (\mathbb{C}^*)^N$ and (2) a toric variety compactifying H.

What happens when we put in a cluster variety? What if we use only $\mathrm{Trop}^+(X)$, which seems to have some much extra structure, rather than $\mathrm{Trop}\, X$?

If we put in $G(2,n)$, then there is a symmetry by a $(n-1)$-dimensional torus H. We have $G(2,n)_0/H \cong M_{0,n}$, where $G(2,n)_0$ is the open locus in $G(2,n)$ where all the Plucker coordinates are nonzero. We get the compactification $\overline{M}_{0,n}$.

The strata of $\overline{M}_{0,n}$ are indexed by trees with n labelled leaves.



Using $\mathrm{Trop}^+ X$ corresponds to only putting in the strata corresponding to *planar* trees.

S.-Sturmfels and others have studied $\operatorname{Trop} G(d,n)$. Hacking, Keel and Tevelev used these tropical methods to compactify $G(d,n)_0/H$. The moduli space $\overline{M}_{0,n}$ is replaced by Kapranov's Chow quotient $G(d,n)/H$ (or blowups thereof). It parametrizes very stable pairs, objects which they introduced to compactify the moduli space of hyperplane arrangements. For $d = 3$ and n small, we get compactifications of moduli spaces of del Pezzo surfaces.

There is lots of beautiful matroid combinatorics here. Giving a complete description of the resulting space seems impossible, because it seems to require understanding the realizability problem for matroids.

Finally, here is something that's been bugging me about compactifications of cluster varieties. I don't know of anything tropical about it, but it feels related.

Let's start with a cluster algebra with frozen variables y_1, y_2, $\ldots$, y_m. Cluster algebra papers generally require coefficients to be in a semi-field, so the y_i have to be invertible. But there is nothing in the cluster technology which requires this. Let A be the algebra generated by the y_i and all the cluster variables.

Does A/y_i inherit a cluster structure? What can we say about it? What are its frozen variables and what happens when we, in turn, quotient by them.

If A is the homogenous coordinate ring of $G(k,n)$, then the frozen variables are $p_{12\cdots k}$, $p_{23\cdots k(k+1)}$, $\ldots$, $p_{n12\cdots(k-1)}$. If we set one of them equal to 0, we get a Schubert divisor, which also has a cluster structure with n-frozen variables. If we set one of those equal to 0, we get one of Postnikov's positroid varieties, which again have n-frozen variables. Keeping going in this way, we encounter all the positroid varieties and rediscover Postnikov's beautiful theory. (Some details here have not been checked, to my knowledge.)

Something similar happens with double Bruhat cells.

Is there a general theory here?

Thank you for coming to my lectures!