Eigenvectors of a symmetric matrix

We have seen two kinds of bases for $\mathbb{R}^n$ that are particularly convenient:

• Orthonormal bases, where our intuition from Euclidean geometry is relevant

• Eigenbases, which are good for computing powers of matrices.

What happens if we put them together?

Let A be a square matrix. Suppose that A has an orthonormal eigenbasis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. What can we say about A?

Let Q be the matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Let D be the diagonal matrix with diagonal $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then we have

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We deduce that

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A.$$

In other words, A is symmetric!

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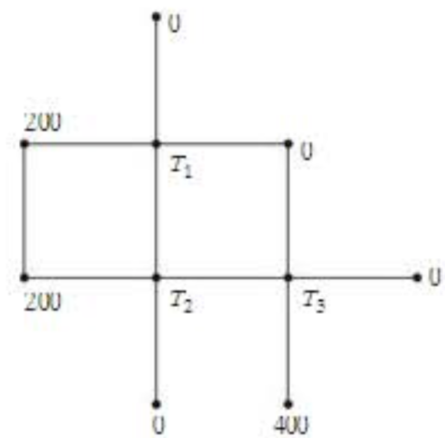
Symmetric matrices come up in many places.

36. A machine contains the grid of wires shown in the accompanying sketch. At the seven indicated points, the temperature is kept fixed at the given values (in °C). Consider the temperatures $T_1(t)$, $T_2(t)$, and $T_3(t)$ at the other three mesh points. Because of heat flow along the wires, the temperatures $T_i(t)$ changes according to the formula

$$T_i(t+1) = T_i(t) - \frac{1}{10}\sum (T_i(t) - T_{\text{adj}}(t)).$$



$$\begin{bmatrix} T_1(t+1) \\ T_2(t+1) \\ T_3(t+1) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{bmatrix}$$

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The big result about symmetric matrices is that the reverse is true:

The spectral theorem: If A is a symmetric $n \times n$ matrix, then A has an orthonormal eigenbasis.

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The easy part of this is that, if A has an eigenbasis, then it has an orthonormal eigenbasis.

Let's see why, if A is a symmetric matrix with an eigenbasis, then A has an orthonormal eigenbasis.

Let $\vec{v}$ and $\vec{w}$ be any two vectors. Since A is symmetric, $\vec{v}^T A\vec{w} = \vec{v}^T A^T \vec{w} = (A\vec{v})^T \vec{w}$. In other words, $\vec{v} \cdot (A\vec{w}) = (A\vec{v}) \cdot \vec{w}$.

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Now, let $\vec{v}$ and $\vec{w}$ be two eigenvectors of A, with distinct eigenvalues α and β.

$$\vec{v}\cdot(\beta\vec{w}) = \vec{w}\cdot(\alpha\vec{v}).$$

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So $\vec{v}$ and $\vec{w}$ are orthogonal! We have shown that any eigenbasis of A will be orthogonal, and we can rescale such a basis to be orthonormal.

The spectral theorem: If A is a symmetric $n \times n$ matrix, then A has an orthonormal eigenbasis.

So far, we have seen that

• If A has an orthonormal eigenbasis, then A is symmetric.

• If A is symmetric and has an eigenbasis, it has an orthonormal eigenbasis.

That symmetric matrices have eigenbases at all is much harder. We'll prove that later, after we've also talked about singular value decomposition.

For now, two other ways to think about the spectral theorem.

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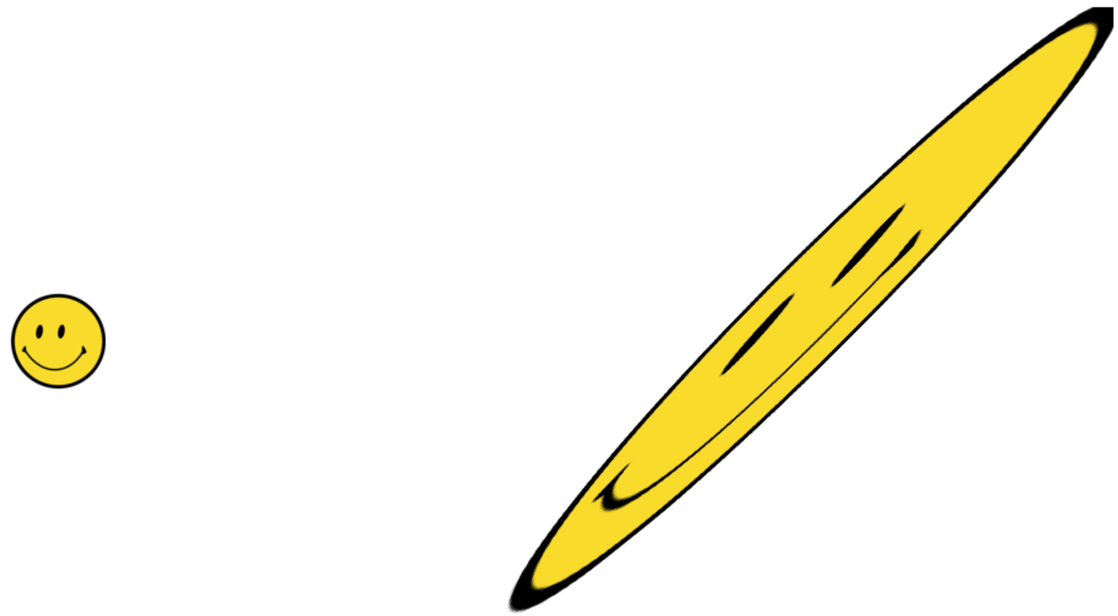

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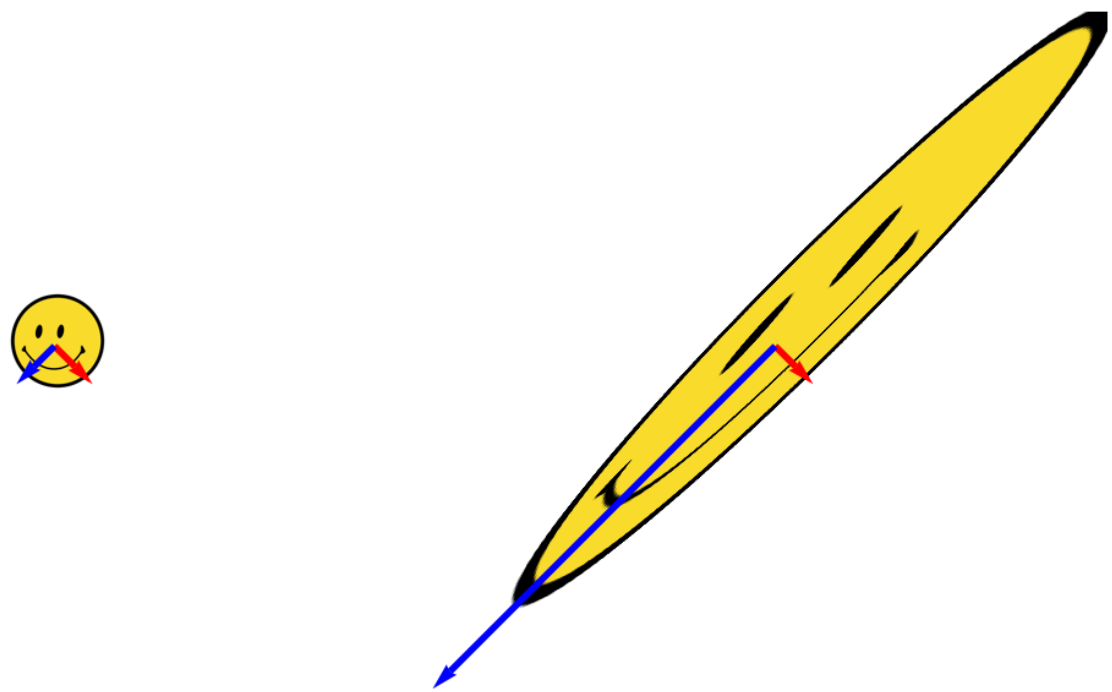

Here is one more important way to think about the eigenvectors of a symmetric matrix. Let A be a symmetric matrix, with orthonormal eigenbasis $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$ and eigenvalues λ_1, λ_2, ..., λ_n. Sort them so that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

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If $\vec{w}$ is a vector of length 1, how long can $A\vec{w}$ be? How short can it be?

Write $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$. The condition that $|\vec{w}| = 1$ is that $c_1^2 + c_2^2 + \cdots + c_n^2 = 1$. We have

$$\begin{aligned}
(A\vec{w}) \cdot (A\vec{w}) &= (c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n) \cdot (c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n) \\
&= c_1^2\lambda_1^2 + \cdots + c_n^2\lambda_n^2
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The largest possible value is to take $c_1 = 1$ and the other to be 0; the smallest is to take $c_n = 1$ and the others to be 0.

So the largest value of $|A\vec{w}|$ is λ_1, by taking $\vec{w} = \vec{v}_1$, and the smallest value of $|A\vec{w}|$ is λ_n, by taking $\vec{w} = \vec{v}_n$.