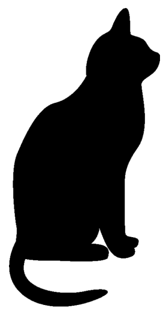
Solving linear equations, and row reduced echelon form

Today we are going to learn an algorithm to put a set of linear equations into a standard form, called ***row reduced echelon form***.

We will use this algorithm for many purposes; for today, let's say that our goal is to solve systems of many linear equations in many variables.

Example: Wanda the witch owns three types of pets – cats, ravens and snakes. She has 10 pets in total, with a total of 16 legs and 4 wings. How many of each type of pet does she own?

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$$
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Subtract $4 \times$ (eq. 1) from (eq. 2).

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Divide (eq. 2) by -2.

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Subtract (eq. 2) from (eq. 1), and subtract $2 \times$ (eq. 2) from (eq. 3).

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$$\begin{array}{rrcrcr} c & & - & s & = & -2 \\ & r & + & 2s & = & 12 \\ & & & s & = & 5 \end{array}$$

Divide (eq. 3) by -4.

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Divide (eq. 3) by -4.

$$\begin{array}{rrrcr} c & & & = & 3 \\ & r & & = & 2 \\ & & s & = & 5 \end{array}$$

Add (eq. 3) to (eq. 1) and subtract twice (eq. 3) from (eq. 2).

Before saying more about this, we introduce shorthand notation – instead of writing a set of equations, we write a matrix.

Instead of $\begin{array}{rcrcrcr} c & + & r & + & s & = & 10 \\ 4c & + & 2r & & & = & 16 \\ & & 2r & & & = & 4 \end{array}$, we write $\left[\begin{array}{ccc|r} 1 & 1 & 1 & 10 \\ 4 & 2 & 0 & 16 \\ 0 & 2 & 0 & 4 \end{array}\right]$.

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Our basic operations are

- Rescaling a row by a constant factor.
- Subtracting a multiple of one row from another row.
- Switching two rows (not seen in our example).

Notice that these operations preserve the solutions to the equations:

$$\left[\begin{array}{ccc|r} 1 & 1 & 1 & 10 \\ 4 & 2 & 0 & 16 \\ 0 & 2 & 0 & 4 \end{array}\right] \leadsto \left[\begin{array}{rrr|r} 1 & 1 & 1 & 10 \\ 0 & -2 & -4 & -24 \\ 0 & 2 & 0 & 4 \end{array}\right] \leadsto \left[\begin{array}{ccc|r} 1 & 1 & 1 & 10 \\ 0 & 1 & 2 & 12 \\ 0 & 2 & 0 & 4 \end{array}\right] \leadsto \cdots$$

We sweep across the equations from left to right, isolating one variable after another. In our example, we were able to isolate all three variables:

$$\begin{array}{ccccc} c & & & = & 3 \\ & r & & = & 2 \\ & & s & = & 5 \end{array} \qquad \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

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More generally, what we can hope for is to put our equations/matrix into ***row reduced echelon form***.

- Either row is either all 0's, or else its first nonzero entry is a 1. This 1 is called a ***pivot***.
- In a column which contains a pivot, called a ***pivot column***, all the other entries are 0.
- The nonzero rows are at the top of the matrix; they are ordered so that the pivots go from left to right as we go down the rows.

Row reduced echelon form

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- The nonzero rows are at the top of the matrix; they are ordered so that the pivots go from left to right as we go down the rows.

$$\begin{bmatrix} \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Putting a matrix into row reduced echelon form

To make a matrix into row reduced echelon form (rref), we work from left to right. Look at the leftmost column which is not yet a pivot column, and which has a nonzero in a non-pivot row.

- Rescale that entry to be 1.
- Subtract appropriate multiples of the row with the 1 from other rows to make the other entries of that column be 0.
- Switch rows, if needed, to put that row immediately below the already existing pivot rows.

$$\begin{bmatrix} 3 & 6 & 3 & 12 \\ 1 & 2 & 4 & 13 \\ 2 & 4 & 4 & 14 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 4 & 13 \\ 2 & 4 & 4 & 14 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 2 & 1 & 4 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \boxed{1} & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Vocabulary related to rref

- The initial leading 1's are called ***pivots***. The columns that contain them are called ***pivot columns***; the corresponding variables in our system of equations are called ***pivot variables***.
- The columns/variables which are not pivot columns/variables are called ***free columns/variables***.
- The number of pivots is called the ***rank***. This concept will become very useful when we reach Chapter 3, but we'll make you learn the word now.

Other notes

- The algorithm to put a matrix into rref is called the Gauss-Jordan algorithm. Carl Gauss developed a related method in 1795, when he used it to solve 17 equations in 17 variables to compute the orbit of the asteroid Ceres; he published his method in 1809. The method we are teaching was invented by Wilhelm Jordan and Bernard Clasen, independently, in 1888.
- Row reduced echelon form is one of many useful standard forms: row echelon form, column echelon form, QR decomposition, LR decomposition, UDV factorization, Jordan normal form, ... Our textbook/course has made a decision to teach one algorithm and one form for clarity, but you should not be surprised if you can find a better algorithm or form for your particular application.