Matrix operations
Throughout this course, we’ll be doing computations with matrices. We’ll be talking a lot about why matrices work the way they do, but for right now, let’s learn how they work.
A matrix is a rectangular grid of numbers.

\[
\begin{bmatrix}
7 & 5 & -1 & 12 & 0 \\
3 & 4 & -2 & 0 & 0 \\
0 & -9 & 3 & 10 & 0
\end{bmatrix}
\]
A *row* of a matrix is a horizontal band of entries; a *column* is a vertical band.

\[
\begin{bmatrix}
7 & 5 & -1 & 12 & 0 \\
3 & 4 & -2 & 0 & 0 \\
0 & -9 & 3 & 10 & 0
\end{bmatrix}
\begin{array}{l}
\text{Row 1} \\
\text{Row 2} \\
\text{Row 3}
\end{array}
\]

<table>
<thead>
<tr>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
<th>Column 4</th>
<th>Column 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td>↓</td>
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</tr>
<tr>
<td>[7 \quad 5 \quad -1 \quad 12 \quad 0]</td>
<td>[3 \quad 4 \quad -2 \quad 0 \quad 0]</td>
<td>[0 \quad -9 \quad 3 \quad 10 \quad 0]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
We call a matrix $m \times n$ where $m$ is the number of rows and $n$ is the number of columns.

This is a $3 \times 5$ matrix:

$$
\begin{bmatrix}
7 & 5 & -1 & 12 & 0 \\
3 & 4 & -2 & 0 & 0 \\
0 & -9 & 3 & 10 & 0
\end{bmatrix}
$$

We label the entries of a matrix $A$ as $A_{ij}$ where $i$ is the row label and $j$ is the column label:

$$
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35}
\end{bmatrix}
$$
We can only add two matrices when they have the same size. In that case, we just add their entries:

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23}
\end{bmatrix}
+ 
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23}
\end{bmatrix}
= 
\begin{bmatrix}
A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\
A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23}
\end{bmatrix}.
\]
We multiply two matrices when the number of columns of the first matrix equals the number of rows of the second matrix.
We multiply two matrices when the number of **columns** of the first matrix equals the number of **rows** of the second matrix.

I like to think of a matrix as a machine having output pipes hanging off the lefthand side, and input pipes coming in from the right through the top:

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34}
\end{bmatrix}
\]
We multiply two matrices when the number of **columns** of the first matrix equals the number of **rows** of the second matrix.

We can multiply two matrices when the number of input pipes of the first matrix matches the number of outputs of the second:

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34}
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32} \\
B_{41} & B_{42}
\end{bmatrix}
\]
We multiply two matrices when the number of columns of the first matrix equals the number of rows of the second matrix.

We can multiply two matrices when the number of input pipes of the first matrix matches the number of outputs of the second:

The product of an $\ell \times m$ matrix by a $m \times n$ matrix is an $\ell \times n$ matrix. In other words, jam together the pipes.

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34}
\end{bmatrix}
\times
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32} \\
B_{41} & B_{42}
\end{bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22} \\
C_{31} & C_{32}
\end{bmatrix}
\]
To compute entry \((i, k)\) of the product, add up the products going across the \(i\)-th row of the first matrix and down the \(k\)-th column of the second matrix:

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34}
\end{bmatrix}
\times
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32} \\
B_{41} & B_{42}
\end{bmatrix}
= 
\begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22} \\
C_{31} & C_{32}
\end{bmatrix}
\]

\[C_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} + A_{24}B_{41}.\]

If you like letters better than numbers,

\[C_{ik} = \sum_{j} A_{ij} B_{jk}.\]

Why? See the next lecture, and basically the whole course!
It is important to note that it matters what order we multiply matrices in!

If $AB$ is defined, the product $BA$ need not even be defined:

$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix} \quad \begin{bmatrix} * \\ * \end{bmatrix} \begin{bmatrix} * & * & * \\ * \end{bmatrix} \text{ doesn’t make sense!}$$

Even if $AB$ and $BA$ are defined, they needn’t have the same size:

$$\begin{bmatrix} * \end{bmatrix} \begin{bmatrix} * & * \\ * \end{bmatrix} = \begin{bmatrix} * & * \\ * \end{bmatrix} \quad \begin{bmatrix} * & * \\ * \end{bmatrix} \begin{bmatrix} * \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}.$$

And, even if $AB$ and $BA$ have the same size, they needn’t be equal:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} \quad \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aw+cx & bw+dx \\ ay+cz & by+dz \end{bmatrix}.$$
Properties of matrix operations

I told that, with matrices, $A \times B$ need not be the same thing as $B \times A$. In my experience, most students get used to keeping track of this issue while being comfortable that matrix addition and multiplication "mostly" work the way you expect.

But a few students want a detailed list of exactly which things still work. The point of this lecture is to give you that list.

If you like the idea of studying which properties hold and do not hold for a new type of mathematical object, and the consequences of that, you want to study *Abstract Algebra*. Take Math 312 or 412.
Properties of matrix operations

Whenever the following equations make sense, they are true:

\[ A + 0 = 0 + A = A \quad \text{Id} \ X = X \quad X \ \text{Id} = X \]
\[ A0 = 0 \quad 0A = 0 \]
\[ A + B = B + A \]
\[ (A + B) + C = A + (B + C) \quad (AB)C = A(BC) \]
\[ A(B + C) = AB + AC \quad (X + Y)Z = XZ + YZ \]

where
\[
0 = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\quad \text{and} \quad
\text{Id} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix}
\]
The **transpose** of a matrix is that matrix with the rows and columns switched. We write $A^T$ for the transpose of $A$. For example,

$$
\begin{bmatrix}
a & b & c \\
d & e & f
\end{bmatrix}^T = \begin{bmatrix}
a & d \\
b & e \\
c & f
\end{bmatrix}
$$

We have $(AB)^T = B^T A^T$.

Transpose will really come into its own in Chapter 5, but the book introduces it now, so we will too. Also, there is one very small thing it is useful for immediately . . .
Transpose

We (and your textbook, and most of the world) will generally write vectors as column matrices: So a three dimensional vector would be $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We usually put an arrow over vectors, like this $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

This makes most formulas more convenient, but makes one thing less convenient: Vectors with lots of entries take up a lot of space on the page:

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7 \\
  x_8 \\
  x_9 \\
  x_{10}
\end{bmatrix}.
$$

We can use transpose to solve this notational issue, and write

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7 \\
  x_8 \\
  x_9 \\
  x_{10}
\end{bmatrix}^T
$$

instead.