Linear transformations and geometry
Last, we learned about the concept of linear maps:

A function $f : \mathbb{R}^p \to \mathbb{R}^q$ is called \textit{linear} if

- $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$, for all vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^p$ and
- $f(a\vec{x}) = af(\vec{x})$ for all scalars $a$ and all vectors $\vec{p}$ in $\mathbb{R}$. 
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And we learned that linear maps are given by matrices:

**Theorem** The linear functions from $\mathbb{R}^p$ to $\mathbb{R}^q$ are precisely given by $q \times p$ matrices; they are the functions $f(\vec{x}) = A\vec{x}$ for $A$ a $q \times p$ matrix.
Last, we learned about the concept of linear maps:

A function \( f : \mathbb{R}^p \rightarrow \mathbb{R}^q \) is called **linear** if

- \( f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}) \), for all vectors \( \vec{x} \) and \( \vec{y} \) in \( \mathbb{R}^p \) and
- \( f(a\vec{x}) = af(\vec{x}) \) for all scalars \( a \) and all vectors \( \vec{x} \) in \( \mathbb{R} \).

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Last time, we talked about how linear maps show up in modeling real world scenarios. This time, we are going to talk about linear maps in geometry.
Here is an image, sitting in $\mathbb{R}^2$. 
Here is that image, transformed by a linear map.
We’ll put the origin of our coordinates at the lower left window. We’ll take the blue arrow to be \([\frac{1}{0}]\) and the maize arrow to be \([\frac{0}{1}]\). The images of those arrows in the new image are \([\frac{1.5}{0.5}]\) and \([\frac{0.4}{0.8}]\).
We’ll put the origin of our coordinates at the lower left window. We’ll take the blue arrow to be \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and the maize arrow to be \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). The images of those arrows in the new image are \( \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} \) and \( \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \).

What are the coordinates of the green vector? What are the coordinates of its image in the transformed image?
We’ll put the origin of our coordinates at the lower left window. We’ll take the blue arrow to be \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and the maize arrow to be \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). The images of those arrows in the new image are \( \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} \) and \( \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \). What is the matrix of the linear map?

\[
\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]
In general, suppose we have a linear map

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix} \rightarrow
\begin{bmatrix}
  A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
  A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\
  A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix}.
$$

We can read off the columns of the matrix by looking at the images of the coordinate vectors.
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\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{bmatrix}.
\]

We can read off the columns of the matrix by looking at the images of the coordinate vectors.

\[
A \begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} = \begin{bmatrix}
  A_{11}
\end{bmatrix}, \quad
A \begin{bmatrix}
  0 \\
  1 \\
  0 \\
  0 \\
  0
\end{bmatrix} = \begin{bmatrix}
  A_{21}
\end{bmatrix}, \quad
\text{etcetera}
\]
We’ll put the origin of our coordinates at the lower left window. We’ll take the blue arrow to be \([\frac{1}{0}]\) and the maize arrow to be \([\frac{0}{1}]\). The images of those arrows in the new image are \([\frac{1.5}{0.5}]\) and \([\frac{0.4}{0.8}]\).

The linear map is \([x, y] \mapsto \begin{bmatrix} 1.5 & 0.4 \\ 0.5 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\).