Subspaces
In the previous lecture, we learned about the kernel and image of a linear map.

- The kernel of $A$ is the set of vectors $\vec{x}$ with $A\vec{x} = \vec{0}$.
- The image of $A$ is the set of vectors $\vec{b}$ for which $A\vec{x} = \vec{b}$ is solvable.
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Both of these are examples of *subspaces*. A *subspace* of $\mathbb{R}^n$ is a set $V$ of vectors in $\mathbb{R}^n$ such that

- $\vec{0}$ is in $V$
- If $\vec{x}$ and $\vec{y}$ are in $V$, then $\vec{x} + \vec{y}$ is in $V$
- If $\vec{x}$ is in $V$, and $k$ is a scalar, then $k\vec{x}$ is in $V$. 
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Which of these are subspaces of $\mathbb{R}^2$?
Let’s see why kernel is a subspace:

- We have $A\vec{0} = \vec{0}$.
- If $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$, then $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$.
- If $A\vec{x} = \vec{0}$, then $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$.
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And the same for image:

- We have \( A\vec{0} = \vec{0} \).
- If \( A\vec{x} = \vec{b} \) and \( A\vec{y} = \vec{c} \), then \( A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{b} + \vec{c} \).
- If \( A\vec{x} = \vec{b} \), then \( A(k\vec{x}) = k(A\vec{x}) = k\vec{b} \).
One of the main ways we describe a subspace is as the *span* of a list of vectors.

The span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is the set of all vectors which can be written in the form

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k.$$
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This is also the image of the matrix with columns $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_k$, since

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k$$
The same subspace can be written as the span of many different sets of vectors.

For example, consider the plane $x + y + z = 0$ in $\mathbb{R}^3$. It is described as all of:

$$\text{Span} \left( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right), \text{Span} \left( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right), \text{Span} \left( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right),$$

$$\text{Span} \left( \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right), \ldots$$

This is the theme we’ll pick up in the next lecture.