Eigenbases
Let $A$ be an $n \times n$ square matrix. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be nonzero eigenvectors of $A$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We saw before that, if $\vec{w}$ is in the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, then we have a simple formula for $A^n \vec{w}$. 
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$$A^n (c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k) = c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n \vec{v}_2 + \cdots + c_k \lambda_k^n \vec{v}_k.$$
Let $A$ be an $n \times n$ square matrix. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be nonzero eigenvectors of $A$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We saw before that, if $\vec{w}$ is in the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, then we have a simple formula for $A^n \vec{w}$.

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So it is useful to express vectors as linear combinations of eigenvectors. The ideal situation would be is $A$ has a basis of eigenvectors. We call this an *eigenbasis*. 
Let $A$ be an $n \times n$ square matrix. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be nonzero eigenvectors of $A$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Let’s talk about whether or not $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ could be a basis.

There are two issues:

1. Are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ linearly independent?

2. Do $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span $\mathbb{R}^n$?
Are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ linearly independent?

The first question has a very clear answer!

**Theorem:** Let $A$ be an $n \times n$ square matrix. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are scalars, no two of them equal, and that $\vec{v}_i$ is a nonzero $\lambda_i$-eigenvector of $A$. Then the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.
Are \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) linearly independent?

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Let’s first see why it works for \( k = 2 \).
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Let’s first see why it works for $k = 2$. To say that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent is to say that they are both nonzero, and they are not proportional.

We assumed that $\vec{v}_1$ and $\vec{v}_2$ are nonzero. Suppose that they were proportional, so that $\vec{v}_2 = c\vec{v}_1$. Then $A\vec{v}_2 = A(c\vec{v}_1) = cA\vec{v}_1 = c\lambda_1 \vec{v}_1$ and also $A\vec{v}_2 = \lambda_2 \vec{v}_2 = c\lambda_2 \vec{v}_1$. 
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So $c\lambda_1 \vec{v}_1 = c\lambda_2 \vec{v}_1$ and $c(\lambda_1 - \lambda_2)\vec{v}_1 = 0$. But $\lambda_1 - \lambda_2$ is not zero, since the lambdas are different, the scalar $c$ is nonzero since $\vec{v}_2 \neq 0$, and the vector $\vec{v}_1$ is also nonzero. This gives a contradiction.
**Theorem:** Let $A$ be an $n \times n$ square matrix. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are scalars, no two of them equal, and that $\vec{v}_i$ is a nonzero $\lambda_i$-eigenvector of $A$. Then the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

We will prove the general case by induction on $k$. So, suppose we have already shown that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}$ are linearly independent. We want to show that adding in the vector $\vec{v}_k$ doesn’t disturb the linear independence. Suppose, for the sake of contradiction, that $\vec{v}_k$ is redundant. So

$$\vec{v}_k = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_{k-1} \vec{v}_{k-1}.$$
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We have:

$$A\vec{v}_k = A(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_{k-1} \vec{v}_{k-1})$$

$$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \cdots + c_{k-1} A\vec{v}_{k-1}$$

$$= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \cdots + c_{k-1} \lambda_{k-1} \vec{v}_{k-1}.$$  

But also:

$$A\vec{v}_k = \lambda_k \vec{v}_k$$

$$= \lambda_k (c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_{k-1} \vec{v}_{k-1})$$

$$= c_1 \lambda_k \vec{v}_1 + c_2 \lambda_k \vec{v}_2 + \cdots + c_{k-1} \lambda_k \vec{v}_{k-1}.$$  

So $c_1 \lambda_1 \vec{v}_1 + \cdots + c_{k-1} \lambda_{k-1} \vec{v}_{k-1} = c_1 \lambda_k \vec{v}_1 + \cdots + c_{k-1} \lambda_k \vec{v}_{k-1}$

and $c_1 (\lambda_1 - \lambda_k) \vec{v}_1 + \cdots + c_{k-1} (\lambda_{k-1} - \lambda_k) \vec{v}_{k-1} = \vec{0}.$
Suppose we have already shown that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}$ are linearly independent. Suppose, for the sake of contradiction, that $\vec{v}_k$ is redundant. So

$$\vec{v}_k = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_{k-1} \vec{v}_{k-1}$$

$$\cdots \text{after some computations}\cdots$$

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But we assumed that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}$ were linearly independent, so this means that

$$c_1(\lambda_1 - \lambda_k) = c_2(\lambda_2 - \lambda_k) = \cdots = c_{k-1}(\lambda_{k-1} - \lambda_k) = 0.$$ 

And the $\lambda$’s are all different, so $c_1 = c_2 = \cdots = c_{k-1} = 0.$
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And the $\lambda$’s are all different, so $c_1 = c_2 = \cdots = c_{k-1} = 0$.

But then $\vec{v}_k = 0 + 0 + \cdots + 0 = 0$, contradicting that the $\vec{v}_j$ are nonzero eigenvectors. $\textbf{QED}$
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   ○ To discuss the general case, we introduce the notions of geometric and algebraic multiplicity. These are the topic of the next lecture.