Algebraic and Geometric Multiplicity

Let A be an $n \times n$ square matrix. We learned how to compute the eigenvalues and eigenvectors of A:

1. Compute the characteristic polynomial, $\det(A - t\mathrm{Id})$, and find its roots. These are the **eigenvalues**.
2. For each eigenvalue λ, compute $\mathrm{Ker}(A - \lambda\mathrm{Id})$. This is the **$\lambda$-eigenspace**, the vectors in the **λ-eigenspace** are the **λ-eigenvectors**.

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We learned that it is particularly nice when A has an **eigenbasis**, because then we can **diagonalize** A. An eigenbasis is a basis of eigenvectors.

Let's see what can happen when we carry out this algorithm.

The simplest case: The characteristic polynomial has n distinct roots

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\det(A - t\mathrm{Id}) = \det \begin{bmatrix} 4-t & 1 \\ -2 & 1-t \end{bmatrix} =$$

$$(4-t)(1-t)-(-2) = (t^2-5t+4)+2 = t^2-5t+6 = (t-2)(t-3).$$

2-eigenspace: $\mathrm{Ker}\,(A - 2\mathrm{Id}) = \mathrm{Ker}\left[\begin{smallmatrix} 4-2 & 1 \\ -2 & 1-2 \end{smallmatrix}\right] = \mathrm{Ker}\left[\begin{smallmatrix} 2 & 1 \\ -2 & -1 \end{smallmatrix}\right]$ The kernel is $\mathrm{Span}\left(\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]\right)$.

3-eigenspace: $\mathrm{Ker}\,(A - 3\mathrm{Id}) = \mathrm{Ker}\left[\begin{smallmatrix} 4-3 & 1 \\ -2 & 1-3 \end{smallmatrix}\right] = \mathrm{Ker}\left[\begin{smallmatrix} 1 & 1 \\ -2 & -2 \end{smallmatrix}\right]$ The kernel is $\mathrm{Span}\left(\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]\right)$.

The simplest case: The characteristic polynomial has n distinct roots

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \qquad \lambda_1 = 2,\ \vec{v}_1 = \left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right] \qquad \lambda_2 = 3,\ \vec{v}_2 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right].$$

The vectors $\vec{v}_1$ and $\vec{v}_2$ form a basis. There are n of them, and λ_1 and λ_2 are distinct.

The characteristic polynomial may not have roots

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

$$\det(A - t\mathrm{Id}) = \det \begin{bmatrix} -t & 1 \\ -2 & -t \end{bmatrix} = t^2 - (-2) = t^2 + 2.$$

No eigenvalues!

Or, at least, not real ones. We'll talk about complex numbers in the next lecture.

The characteristic polynomial can have repeated roots

Here are two examples:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad \det(A. - t\mathrm{Id}) = \det \begin{bmatrix} 3-t & 0 \\ 0 & 3-t \end{bmatrix} = (3-t)^2.$$

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad \det(B. - t\mathrm{Id}) = \det \begin{bmatrix} 3-t & 1 \\ 0 & 3-t \end{bmatrix} = (3-t)^2.$$

$$\mathrm{Ker}(A - 3\mathrm{Id}) = \mathrm{Ker} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{R}^2.$$

$$\mathrm{Ker}(B - 3\mathrm{Id}) = \mathrm{Ker} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{Span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

A and B have the same characteristic polynomial: $(3-t)^2$. But A has an eigenbasis – any basis for $\mathbb{R}^2$ is an eigenbasis – and B doesn't.

We introduce two vocabulary words to describe what we have seen. Let A be a square matrix and λ a scalar.

- ***The geometric multiplicity of*** λ is the dimension of the λ-eigenspace. In other words, $\dim \operatorname{Ker}(A - \lambda \mathrm{Id})$.

- ***The algebraic multiplicity of*** λ is the number of times $(\lambda - t)$ occurs as a factor of $\det(A - t\mathrm{Id})$.

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For example, take $B = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$. Then $\operatorname{Ker}(B - 3\mathrm{Id}) = \operatorname{Ker}\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is one dimensional, so the geometric multiplicity is 1.

But $\det(B - t\mathrm{Id}) = \det\left[\begin{smallmatrix} 3-t & 1 \\ 0 & 3-t \end{smallmatrix}\right] = (3-t)^2$, so the algebraic multiplicity is 2.

By contrast, for the matrix $A = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$, both the algebraic and geometric multiplicity are 2.

Let A be an $n \times n$ square matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Let a_i be the algebraic multiplicity of λ_i. Let V_i be the λ_i-eigenspace and let $g_i = \dim V_i$, the geometric multiplicity of λ_i.

Theorem 1: We have $\sum g_i = n$ if and only if A has an eigenbasis.

So Theorem 1 gives us an answer to when we have an eigenbasis. But it requires computing eigenspaces. What can we see from just the characteristic polynomial?

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Theorem 2: We have $g_i \leq a_i$.

So $\sum g_i \leq \sum a_i$. But we can't compute g_i from a_i: The matrices $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ both have algebraic multiplicity 2, but one has geometric multiplicity 2 and the other has geometric multiplicity 1.

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Theorem 3: We have $\sum a_i \leq n$, and $\sum a_i = n$ if and only if $\det(A - t\mathrm{Id})$ factors completely into linear terms.

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So we have an eigenbasis if and only if $g_i = a_i$ and $\det(A - t\mathrm{Id})$ factors completely into linear terms.

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...to be continued.