Algebraic and Geometric Multiplicity: Proofs

Let A be an $n \times n$ square matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Let a_i be the algebraic multiplicity of λ_i. Let V_i be the λ_i-eigenspace and let $g_i = \dim V_i$, the geometric multiplicity of λ_i.

Theorem 1: We have $\sum g_i = n$ if and only if A has an eigenbasis.

Theorem 2: We have $g_i \leq a_i$.

Theorem 3: We have $\sum a_i \leq n$, and $\sum a_i = n$ if and only if $\det(A - t\mathrm{Id})$ factors completely into linear terms.

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We told the story in the order that explains the importance of the results, but the proofs are easiest in the reverse order.

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We have

$$\det(A - t\mathrm{Id}) = \prod_i (\lambda_i - t)^{a_i} \cdot (\text{some polynomial with no roots}).$$

So

$$\deg \det(A - t\mathrm{Id}) = n \geq \sum a_i.$$

We have equality if and only if $\det(A - t\mathrm{Id})$ factors into linear terms. **QED**

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To make notation simpler, abbreviate λ_i to λ, g_i to g and a_i to a.

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Complete $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_g$ to a basis
$\mathfrak{B} = (\vec{v}_1,\ \vec{v}_2,\ \ldots, \vec{v}_g,\ \vec{w}_1,\ \vec{w}_2,\ \ldots, \vec{w}_{n-g})$ for $\mathbb{R}^n$.

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$$\left[\begin{array}{cccc|c} \lambda & & & & \\ & \lambda & & & ? \\ & & \ddots & & \\ & & & \lambda & \\ \hline & & 0 & & ? \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} \lambda & & & & \\ & \lambda & & & B \\ & & \ddots & & \\ & & & \lambda & \\ \hline & & 0 & & C \end{array}\right]$$

So

$$A - t\mathrm{Id}_n = \left[\begin{array}{cccc|c} \lambda-t & & & & \\ & \lambda-t & & & B \\ & & \ddots & & \\ & & & \lambda-t & \\ \hline & & 0 & & C - t\mathrm{Id}_{n-g} \end{array}\right]$$

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$$\det(A - t\mathrm{Id}_n) = \det \left[\begin{array}{cccc|c} \lambda-t & & & & \\ & \lambda-t & & & B \\ & & \ddots & & \\ & & & \lambda-t & \\ \hline & & 0 & & C - t\mathrm{Id}_{n-g} \end{array}\right]$$

$$= (\lambda - t)^g \det(C - t\mathrm{Id}_{n-g}). \quad \textbf{QED}$$

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First of all, suppose that A has an eigenbasis: $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$.

For each λ_i, let h_i be the number of $\vec{x}_i$ which are λ_i-eigenvectors. So $n = \sum h_i$.

The λ_i-eigenvectors are contained in V_i, so the number of them is at most $\dim V_i$. So $h_i \leq g_i$.

So $n = \sum h_i \leq \sum g_i$. But we also already noted that $\sum g_i \leq \sum a_i \leq n$. So $\sum g_i = n$.

Let A be an $n \times n$ square matrix with eigenvalues λ_1, λ_2, $\ldots$, λ_k. Let a_i be the algebraic multiplicity of λ_i. Let V_i be the λ_i-eigenspace and let $g_i = \dim V_i$, the geometric multiplicity of λ_i.

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The notation for this one gets hairy, so let's rename things.

Let's suppose there are 3 eigenvalues: α, β, γ.

Let the eigenspaces be U, V and W.

Let the geometric multiplicities be $a = \dim U$, $b = \dim V$, $c = \dim W$, with $a + b + c = n$.

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Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_a$ be a basis of U; let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_b$ be a basis of V and let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_c$ be a basis of W.

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These are eigenvectors, so our goal will be to show that $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_a, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_b, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_c$ is a basis of $\mathbb{R}^n$.

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Let $\vec{u}_1$, $\vec{u}_2$, $\dots$, $\vec{u}_a$ be a basis of U; let $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_b$ be a basis of V and let $\vec{w}_1$, $\vec{w}_2$, $\dots$, $\vec{w}_c$ be a basis of W. Our goal is to show that $\vec{u}_1$, $\vec{u}_2$, $\dots$, $\vec{u}_a$, $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_b$, $\vec{w}_1$, $\vec{w}_2$, $\dots$, $\vec{w}_c$ is a basis of $\mathbb{R}^n$.

Since $a + b + c = n$, we just need to show that $\vec{u}_1$, $\vec{u}_2$, $\dots$, $\vec{u}_a$, $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_b$, $\vec{w}_1$, $\vec{w}_2$, $\dots$, $\vec{w}_c$ are linearly independent.

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Suppose that we had a linear relation:

$$f_1\vec{u}_1 + \cdots + f_a\vec{u}_a + g_1\vec{v}_1 + \cdots + g_b\vec{v}_b + h_1\vec{w}_1 + \cdots + h_c\vec{w}_c = \vec{0}.$$

Let's suppose there are 3 eigenvalues: α, β, γ. Let the eigenspaces be U, V and W. Let the geometric multiplicities be $a = \dim U$, $b = \dim V$, $c = \dim W$, with $a + b + c = n$.

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$$\overbrace{f_1\vec{u}_1 + \cdots + f_a\vec{u}_a}^{\alpha-\text{eigenvector}} + \overbrace{g_1\vec{v}_1 + \cdots + g_b\vec{v}_b}^{\beta-\text{eigenvector}} + \overbrace{h_1\vec{w}_1 + \cdots + h_c\vec{w}_c}^{\gamma-\text{eigenvector}} = \vec{0}.$$

Let's suppose there are 3 eigenvalues: α, β, γ. Let the eigenspaces be U, V and W. Let the geometric multiplicities be $a = \dim U$, $b = \dim V$, $c = \dim W$, with $a + b + c = n$.

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Since α, β and γ are distinct, we must have

$$f_1\vec{u}_1 + \cdots + f_a\vec{u}_a = g_1\vec{v}_1 + \cdots + g_b\vec{v}_b = h_1\vec{w}_1 + \cdots + h_c\vec{w}_c = \vec{0}.$$

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But, since $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_a$ is a basis of U, this means that $f_1 = \cdots = f_a = 0$.

Similarly, $g_1 = \cdots = g_b = 0$ and $h_1 = \cdots = h_c = 0$.

We have shown that all the coefficients of our linear relation are 0. So we have shown that the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_a, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_b, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_c$ are linearly independent.

Since $a + b + c = n$, we have shown that $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_a, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_b, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_c$ is a basis of $\mathbb{R}^n$. **QED**