11. Algebraic elements, minimal polynomials

Let k be a field, and let L be a larger field containing k. Let $\theta \in L$. We say that θ is ***algebraic over*** k if there is a nonzero polynomial $f(x) = f_n x^n + \cdots + f_1 x + f_0$ with $f(\theta) = 0$.

Problem 11.1. Let $\eta = \frac{1+\sqrt{-3}}{2}$, so η is a primitive 6-th root of unity.

(1) Let $a(x) = x^3 + 1$ and let $b(x) = x^4 + x^2 + 1$. Show that $a(\eta) = b(\eta) = 0$.
(2) Of all polynomials $f(x)$ in $\mathbb{Q}[x]$ with $f(\eta) = 0$, which is the polynomial with lowest degree?

Problem 11.2. Let k be a field, let L be a larger field containing k and let θ be algebraic over k. Show that there is a polynomial $m(x)$ in $k[x]$ such that, for all $f(x)$ in $k[x]$, we have $f(\theta) = 0$ if and only if $m(x)$ divides $f(x)$.

Problem 11.3. With the same notation as in the previous problem, show that $m(x)$ is irreducible.

The polynomial $m(x)$ is called the ***minimal polynomial of*** θ.

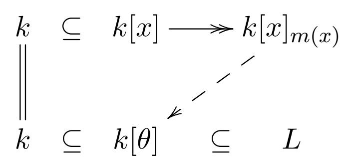
Let $k[\theta]$ be those elements in L which can be written in the form $\sum a_i \theta^i$ for coefficients a_i in k. This notation will come back frequently, and we'll also write things like $k[\theta_1, \theta_2, \ldots, \theta_r]$ when we have more than one element of L.

Problem 11.4. Once again, let η be the primitive 6-th root of unity, $\eta = \frac{1+\sqrt{-3}}{2}$. Write the following numbers in the form $a + b\eta$, with a and b rational:

$$(\eta + 1)^2 \qquad \frac{1}{1-\eta}.$$

Problem 11.5. Let k be a field, let L be a larger field containing k and let θ in L be algebraic over k. Let $m(x)$ be the minimal polynomial of θ over k.

(1) Show that there is a unique map $k[x]_{m(x)} \to k[\theta]$, sending each element of k to itself and sending x to θ.



(2) Show that the map $k[x]_{f(x)} \to k[\theta]$ is a bijection.
(3) Show that $k[\theta]$ is a field.