13. Field homomorphisms and isomorphisms

Let K and L be two fields. A ***homomorphism*** from K to L is a map $\phi : K \to L$ such that

$$\phi(1) = 1,\ \phi(x+y) = \phi(x) + \phi(y) \text{ and } \phi(xy) = \phi(x)\phi(y).$$

A homomorphism is called an ***isomorphism*** if it is a bijection.

Problem 13.1. Let K and L be two fields and let $\phi : K \to L$ be a homomorphism. Show that

(1) We have $\phi(0) = 0$.
(2) We have $\phi(-x) = -\phi(x)$.
(3) If x is a nonzero element of K, then $\phi(x) \neq 0$ and $\phi(x^{-1}) = \phi(x)^{-1}$.

Problem 13.2. Let K and L be two fields and let $\phi : K \to L$ be a homomorphism. Show that ϕ is injective, meaning that, if $\phi(x) = \phi(y)$ then $x = y$.

If $\phi : k_1 \to k_2$ is a homomorphism of fields, and $f(x) = f_n x^n + \cdots + f_1 x + f_0$ is a polynomial in $k[x]$, we write $\phi(f)(x)$ for the polynomial $\phi(f_n)x^n + \cdots + \phi(f_1)x + \phi(f_0)$ in $k_2[x]$.

Problem 13.3. Let $\phi : k_1 \to k_2$ is a homomorphism of fields, $f(x) = f_n x^n + \cdots + f_1 x + f_0$ be a polynomial in $k_1[x]$ and let θ be a root of f in k_1. Show that $\phi(\theta)$ is a root of $\phi(f)$ in k_2.

The following problem is almost identical to Problem 11.5, but it has an isomorphism instead of an equality:

Problem 13.4. Let k_1 and k_2 be two fields and let $\phi : k_1 \to k_2$ be an isomorphism. Let L be a larger field containing k_2 and let θ in L be algebraic over k_2. Let $m(x)$ be the minimal polynomial of θ over k_2. Show that there is a unique isomorphism $k_1[x]_{\phi^{-1}(m)(x)} \to k_2[\theta]$ sending x to θ and making the following diagram commute:

$$\begin{array}{ccccc}
k_1 & \subseteq & k_1[x] & \twoheadrightarrow & k_1[x]_{\phi^{-1}(m)(x)} \\
\downarrow \phi & & & \swarrow & \\
k_2 & \subseteq & k_2[\theta] & \subseteq & L
\end{array}$$

Problem 13.5. Let k_1 and k_2 be two fields and let $\phi : k_1 \to k_2$ be an isomorphism. Let $f(x)$ be a nonzero polynomial in $k_1[x]$. Let L_1 be a field containing k_1, where f has roots $\rho_1, \rho_2, \ldots, \rho_\ell$. Let L_2 be a field in which $\phi(f)(x)$ splits, with roots $\theta_1, \ldots, \theta_m$. Show that there is a homomorphism $\psi : k[\rho_1, \ldots, \rho_\ell] \to k[\theta_1, \ldots, \theta_m]$ making this diagram commute

$$\begin{array}{ccccc}
k_1 & \subseteq & k_1[\rho_1, \ldots, \rho_\ell] & \subseteq & L_1 \\
\downarrow \phi & & \downarrow \psi & & \\
k_2 & \subseteq & k_2[\theta_1, \ldots, \theta_m] & \subseteq & L_2.
\end{array}$$

The rest of this worksheet is applications of Problem 13.5.

Problem 13.6. Let k_1 and k_2 be two fields and let $\phi : k_1 \to k_2$ be an isomorphism. Let $f(x)$ be a nonzero polynomial in $k_1[x]$. Let L_1 be a splitting field for $f(x)$ and let L_2 be a splitting field for $\phi(f)(x)$. Show that there is an **isomorphism** $\psi : L_1 \to L_2$ making the following diagram commute:

$$\begin{array}{ccc}
k_1 & \subseteq & L_1 \\
\downarrow \phi & & \downarrow \psi \\
k_2 & \subseteq & L_2.
\end{array}$$

Problem 13.7. Let k be a field, let f be a nonzero polynomial in $k[x]$ and let L_1 and L_2 be two splitting fields of $f(x)$. Show that L_1 is isomorphic to L_2. So **splitting fields are unique up to isomorphism**.

Problem 13.8. Let k be a field and let $f(x)$ and $g(x)$ be nonzero polynomials in $k[x]$. Let L be a splitting field for $f(x)$ and let M be a splitting field for $f(x)g(x)$. Show that there is an injection $L \hookrightarrow M$.