Problem 16.1. We now consider a chain of fields $F \subseteq K \subseteq L$ where both K and L are splitting fields of polynomials over F.

(1) Show that every automorphism in $\mathrm{Aut}(L/F)$ maps K to itself.
(2) Let ψ be in $\mathrm{Aut}(L/F)$. Show that the restriction of ψ to $\mathrm{Aut}(L/F)$ is an automorphism of K.
(3) Show restriction from L to K gives a group homomorphism $\mathrm{Aut}(L/F) \to \mathrm{Aut}(K/F)$.
(4) Show that the kernel of the restriction map is $\mathrm{Aut}(L/K)$.
(5) Show that the restriction map is surjective. Hint: Look back at Problem 13.6.

The result of Problem 16.1 is important enough that we state it as a Theorem. We first want a definition: A ***short exact sequence of groups*** is three groups A, B, C and two group homomorphisms $\alpha : A \to B$ and $\beta : B \to C$ such that α is injective, β is surjective and $\alpha(A)$ is the kernel of β.

Theorem: Let $F \subseteq K \subseteq L$ be a chain of fields where both K and L are splitting fields of polynomials over F. Then we have a short exact sequence

$$1 \to \mathrm{Aut}(L/K) \to \mathrm{Aut}(L/F) \to \mathrm{Aut}(K/F) \to 1.$$

Describe $\mathrm{Aut}(L/K)$, $\mathrm{Aut}(L/F)$ and $\mathrm{Aut}(K/F)$ for the following examples:

Problem 16.2. Taking $e_1 = r_1 + r_2 + r_3$, $e_2 = r_1 r_2 + r_1 r_3 + r_2 r_3$, $e_3 = r_1 r_2 r_3$:

$$F = \mathbb{C}(e_1, e_2, e_3) \quad \subset \quad K = F((r_1 - r_2)(r_1 - r_3)(r_2 - r_3)) \quad \subset \quad L = F(r_1, r_2, r_3).$$

Problem 16.3.

$$F = \mathbb{Q} \quad \subset \quad K = \mathbb{Q}(\omega) \quad \subset \quad L = \mathbb{Q}(\omega, \sqrt[3]{2}).$$

Problem 16.4. Taking $e_1 = r_1 + r_2 + r_3 + r_4$, etcetera,

$$F = \mathbb{C}(e_1, e_2, e_3, e_4) \quad \subset \quad K = F(r_1 r_2 + r_3 r_4, r_1 r_3 + r_2 r_4, r_1 r_4 + r_2 r_3) \quad \subset \quad L = F(r_1, r_2, r_3, r_4).$$

Problem 16.5. With ζ being a primitive 5-th root of unity:

$$F = \mathbb{Q} \quad \subset \quad K = \mathbb{Q}(\zeta) \quad \subset \quad L = \mathbb{Q}(\zeta, \sqrt[5]{2}).$$

Problem 16.6. With ζ being a primitive 5-th root of unity:

$$F = \mathbb{Q} \quad \subset \quad K = \mathbb{Q}(\zeta - \zeta^2 - \zeta^3 + \zeta^4) \quad \subset \quad L = \mathbb{Q}(\zeta).$$

Bonus fun question: $\zeta - \zeta^2 - \zeta^3 + \zeta^4$ has a much simpler name. Can you figure out what it is?