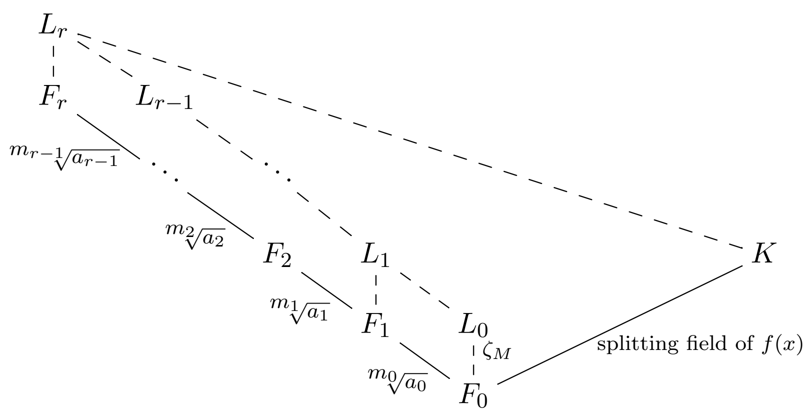
We are finally ready to return to our goal of studying solvability of equations. Suppose that we start with a polynomial $f(x)$ whose coefficients are in a field F_0. We compute a sequence of numbers θ_1, θ_2, θ_3, … where each number is either in our starting field F_0, or is computed from earlier numbers using the functions $+$, $-$, $\times$, $\div$, $\sqrt[m]{}$. Let $F_i = F_0[\theta_1, \theta_2, \ldots, \theta_i]$. So, if θ_i is computed using $+$, $-$, $\times$, $\div$ then $F_i = F_{i-1}$ and, if θ_i is computed using $\sqrt[m]{}$, then $F_i = F_{i-1}[\sqrt[m]{a}]$ for some a in F_{i-1} and some positive integer m. We want to study whether we can ever get to a root of $f(x)$ by this process.

We now state, precisely, what we will prove:

Theorem: Let F_0 be a field, let $f(x)$ be a polynomial with coefficients in F_0 and let K be a splitting field for $f(x)$. Suppose that $\mathrm{Aut}(K/F_0)$ is S_n, for $n \geq 5$. Let F_0, F_1, F_2, …, be a sequence of fields such that each F_{i+1} is of the form $F_i[\sqrt[m_i]{a_i}]$ for some positive integer m_i and some a_i in F_i. Then $f(x)$ does not have roots in any of the F_i.



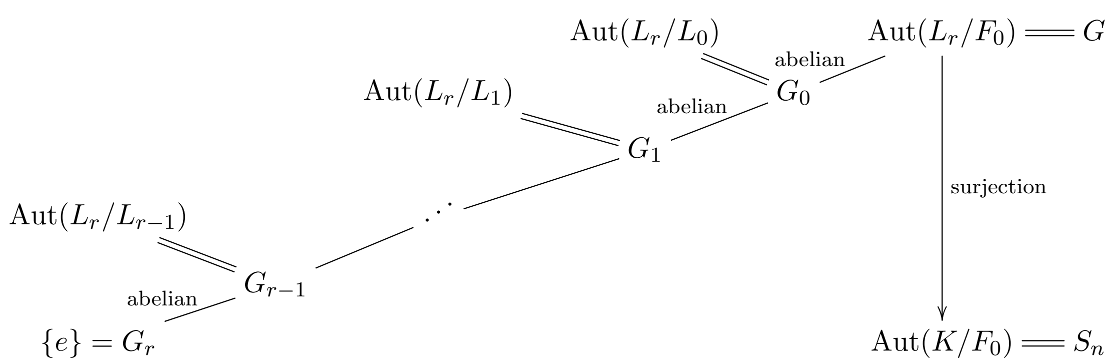
Suppose that we did have fields as above and f that has a root in F_r. In this worksheet, we will introduce a lot of fields. The diagram above shows the relation between them. The solid lines are the hypotheses of the theorem, and the dashed lines are the containments you will be proving:

Let M be the LCM of all the m_i. Let $g_i(x)$ be the minimal polynomial of a_i over F_0. Let L_j be the splitting field of $(x^M - 1)\prod_{i=0}^{j-1} g_i(x^{m_i})$ over F_0. Notice that L_0 is the splitting field of $x^M - 1$ over F_0.

Problem 17.1. (1) Show that we can embed F_i into L_i, as in the diagram.
(2) Show that we can embed K into L_r, as in the diagram.

Problem 17.2. (1) Show that $g_i(x)$ splits in L_i. Let its roots be α_1, α_2, …, α_s.
(2) Show that $L_{i+1} = L_i[\sqrt[m_i]{\alpha_1}, \sqrt[m_i]{\alpha_2}, \cdots, \sqrt[m_i]{\alpha_s}]$.

Inside $\mathrm{Aut}(L_r/F_0)$, we have the subgroups $G_i = \mathrm{Aut}(L_r/L_i)$. Put $\mathrm{Aut}(L_r/F_0) = G$.



Problem 17.3. (1) Show that we have a **surjection** $G \to \mathrm{Aut}(K/F_0)$.
(2) Show that we have a short exact sequence $1 \to G_i \to G_{i-1} \to A_i \to 1$ where A_i is **abelian**.
(3) Show that we have a short exact sequence $1 \to G_0 \to G \to B \to 1$ where B is **abelian**.